II. General Dynamic Problem in Resources, Calculus of Variations

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1) Examples of control problems in natural resources

\( x = \) stock of resources, \( x_0 = \) initial condition
\( y = \) "harvest"

\( f(x) = \) natural growth

\( U(x,y) \) flow of payoff - the integrand

\( r = \) discount rate
\( e^{-rt} = \) discount factor

\( T = \) final time \( x_T = \) terminal condition

\[
\max_{x_0} \int_0^T e^{-rt} U(x(t),y(t)) dt
\]

\[ P1 \]
\[ s.t. \quad \dot{x} = f(x) - y \quad \text{state equation} \]
\[ x_{t0} \text{ given} \quad \text{initial condition} \]
\[ x(T) = x_T \quad \text{terminal condition} \]

In this problem, \( x \) is the state variable, \( y \) is the control variable, and \( \frac{dx}{dt} = f(x) - y \) is the equation of motion.

+ +/- fisheries \( U(x, y) \)
fisheries (compet) \[ = p(t)y - c(x, y) \]
fisheries (monopoly) \[ = p(y) \cdot y - c(x, y) \]
fisheries (social planner) \[ = w(y) - c(x, y) \text{ where } w' = p(y) \]

\[ \dot{x} = f(x) + g \text{ contribution to pollution} \]

\[ U(x, g) = v(g) - D(x) \]

benefit function

\[ U(x, g) = v(g) - D(x) \]

damage function

\[ \dot{x} = -y \]

For the C.O.V. problem write integrand as \( F(t, x, \dot{x}) \) using \( y = f(x) - \dot{x} \). For example, for fishery controlled by social planner:

\[
F(t, x, \dot{x}) \equiv [w(f(x) - \dot{x}) - c(x, f(x) - \dot{x})]e^{-rt}
\]

I can write P1 as

\[
P2 \left\{ \begin{array}{c}
\max_{t_0} \int_0^T F(t, x, \dot{x}) \, dt \\
s.t. \quad x_0 \text{ given} \\
\quad x_T \text{ given}
\end{array} \right.
\]

I have eliminated the constraint \( \dot{x} = f(x) - y \) by substitution. I get rid of one variable (y) and one constraint and write the problem in terms of x and its derivative dx/dt. It is common to eliminate the control variable (here y) when using COV to solve a dynamic problem. That is, instead of choosing the optimal trajectory for the control variable y (harvest), we explicitly choose dx/dt.\(^1\)

2. Necessary Condition

\(^1\)When using the Maximum Principle, we typically retain the control variable y, and we do not eliminate the equation of motion.
(Subscripts indicate partial derivatives.) A necessary condition for P2 is that the optimal $x^*$ solves the Euler equation:

$$F_x(t, x^*, \dot{x}^*) = \frac{d}{dt} \left[ F_x(t, x^*, \dot{x}^*) \right]$$

$$= F_{xt} + F_{xx} \frac{dx}{dt} + F_{x\dot{x}} \frac{d\dot{x}}{dt}$$

$$= F_{xt} + F_{xx} \dot{x} + F_{x\dot{x}} \ddot{x}$$

This is a second order ODE, and we have 2 B.C.'s, $x(0)=x_o$ and $x(T)=x_T$ This is a 2 point boundary value problem (TPBVP)

3. Interpretation of Euler Equation

Interpretation of Euler equation: Integrate, using

$$\int_{t}^{t_1} F_x(t, \tau) d\tau = \int_{t}^{t_1} dF_x(t, \tau) = F_x(t_1) - F_x(t)$$

or,

$$\int_{t}^{t_1} F_x(t, \tau, \dot{x}(\tau), \ddot{x}(\tau)) d\tau = F_x(t_1, x^*(t_1), \dot{x}^*(t_1)) - F_x(t, x^*(t), \dot{x}^*(t))$$

Benefit over $t, t_1$ of having the extra fish in the stock Leaving one more fish in sea at $t$ and taking it out at $t_1$ (reallocating consumption)

Mention case where $F_x \equiv 0$. This inequality would hold if, for example, a larger stock does not make it any cheaper to catch fish, and the growth is independent of the stock of fish. The independence of growth on the stock of fish is not plausible. However, the independence of the growth on the stock does hold for a nonrenewable resource, such as oil, or for a resource with exogenous growth (e.g. the water in an aquifer, where additions to the stock might be independent of the amount of water currently in the aquifer).

If $F_x \equiv 0$ the Euler equation implies that the present value of the marginal utility of consumption is constant.

Continue interpretation of Euler Equation for fish example with no stock dependent costs

$$\dot{x} = f(x) - y$$

2:3
Here $y$ is the harvest (consumption) and $U(y)$ is utility of consuming fish.

Euler equation is

$$ \frac{d}{dt} \left( e^{-rt} U'(y) \right) = e^{-rt} U'(y) f'(x) $$

If I integrate EE over $(t, t_1)$ I get

$$ e^{-rt} U'(y_t) = \int_t^{t_1} e^{-rs} U'(y_s) f'(x_s) ds + e^{-rt_1} U'(y_{t_1}) $$

LHS = PDV of consuming extra fish at $t$

Consider the following perturbation of the optimal trajectory. If I refrained from eating the marginal fish at time $t$, and then ate it at time $t_1$, I can eat the extra fish produced by this marginal fish over the interval $(t, t_1)$. The marginal fish (that I refrain from eating at time $t$) increases the growth by $f'(x)$. When I eat these extra fish, I obtain a PDV of marginal utility of $e^{-rt} U'(y_s) f'(x_s)$ at time $s < t_1$ (and I still have the marginal fish at time $s$). I eat the marginal fish at $t_1$. At optimum, there is no benefit of reallocating consumption in this manner. In other words, the perturbation that I described yields no increase in utility.

In the problem above, time enters only through the discount factor, and because of the upper limit of integration -- the finite horizon -- $T$. This fact makes it possible to eliminate $t$ from the Euler equation, as shown below.

If I write out Euler equation (which requires that I find $ \frac{d}{dt}[e^{-rt} U'(y)]$ and cancel $e^{-rt}$ I have:

$$ re^{-rt} U'(y) - e^{-rt} U''(y) \dot{y} = e^{-rt} U'(y) f'(x) $$

$$ - \frac{U''(y)}{U'(y)} \dot{y} = f'(x) - r $$

This expression is independent of time.
I can write LHS of the last version of the Euler Equation as 
\[
\frac{d}{dt}(U'(y))
\]
In words, the EE says that along the optimal trajectory, the proportionate rate of change of marginal utility must equal the difference between growth rate of capital stock (fish) and "rate of impatience"

\[
\frac{d}{dt}(U'(y)) = \frac{U''}{U'} > 0
\]

\[
\Rightarrow \dot{y} > 0 \quad \text{iff} \quad f'(x) - r > 0
\]

Special case \(U(y) = \ln y\), Euler equation is

\[
\frac{\dot{y}}{y} = f'(x) - r
\]

4. Autonomous Problems and Steady States

Consider the previous fish problem (where harvest costs do not depend on the stock of fish), but replace the upper limit of integration \(T\) by infinity . The optimal trajectory to this infinite horizon problem converges to a steady state. At this steady state (where \(y\) is constant) it must be the case \(f'(x) = r\). In other words, the steady state for the infinite horizon version of this problem is independent of the utility function, \(U\). The steady state requires that the marginal growth rate of fish equals the discount rate.

The control problem is said to be autonomous if and only if the horizon is infinite and time enters explicitly only through a constant discount rate. In this case, the optimal control depends only on the state, not on time.

Think about why this time-independence requires an infinite horizon. If there were a finite horizon, the amount of time to go would affect the optimal control. If there is a finite horizon, but time does not enter the integrand except via discounting, we saw from the previous section that the Euler Equation is independent of time. However, with a finite horizon, the amount of time until we reach the terminal boundary condition is finite, and it changes as calendar time changes. Thus, the solution to the two-point-boundary-value problem depends explicitly on time. In contrast, if the time horizon is infinite, the time to go is always infinite -- i.e. it is constant.
I will consider the (infinite horizon) autonomous problem again, and use the fish example to interpret S.S. condition. The general problem is:

\[
\max_{x} \int_{0}^{\infty} e^{-rt} F(x, \dot{x}) dt \\
\text{s.t. } x(0) = x_0
\]

The Euler Equation is

\[
F_x = -rF_x + F_{xx} \dot{x} + F_{x\dot{x}} \ddot{x}
\]

Notice that I canceled the discount factor in order to write the Euler Equation as independent of time. By definition, variables are constant in steady state. In steady state  \( \dot{x} = \ddot{x} = 0 \). So optimal steady state \( x_s \) solves

\[
(5) \quad F_x(x_s, 0) = -rF_x(x_s, 0)
\]

Interpret (5) for example of fish problem in the general case where the cost of harvesting does depend on the stock of fish -- in contrast to the special case that I considered at the beginning of this section. Divide both sides of (5) by \( r \), and interpret the left side as the PDV of the gain of having one more fish in the stock; the right side equals the cost of reducing consumption by one fish at a point in time. The following discussion explains this interpretation.

As before, the growth equation is

\[
\dot{x} = f(x) - y \\
y = f(x) - \dot{x}
\]

Here I want to allow the function \( F(\cdot) \) -- the integrand -- to include a stock-dependent harvest cost.

\[
F = U(y) \quad B \quad c(x, y) = U(f(x) - x) - c(x, f(x) - x)
\]

I want to interpret the SS condition. Suppose that I am at the steady state and I decide to eat one less fish, and thereby increase the steady state by one fish. The costs and benefits of this marginal change should exactly balance each other at the optimal steady state -- as the Euler Equation implies.

If I'm at the SS and decide to consume one less fish my change in flow of welfare at the time I reduce
my consumption (consume one less fish) is

\[ F_x = -U'(y) + c_z = \]

change in utility of consumption + change in harvest cost.

Having one more fish in the stock increases the flow of future consumption (= growth in steady state) by \( f'(x) \). The change in flow of total welfare equals change in utility from consumption \( U'(y)f'(x) \), minus the change in harvest cost. The harvest cost at the steady state is \( c(x, f(x)) \), so the change in cost due to the change in steady state is \( c_1 + c_2f' \). In symbols:

\[ F_x = U'f' - (c_1 + c_2f') \]

Change in total utility discounted over infinite horizon is \( \frac{F_x}{r} \).

Exercise. Consider the case where growth function \( f(x) \) is "inverted U". The level of stock that provides maximum sustainable yield solves \( f'(x) = 0 \). What factors determine whether the socially optimal steady state is above or below the level that maximizes sustainable yield?

Comment: the necessary condition using COV is a second order ODE. For the autonomous problem (no explicit dependence on time except via constant discounting, and infinite horizon) we can use phase plane analysis to analyze the solution. In order to do this, we need to convert the second order ODE (the EE) into a system of two first order ODEs. For example, if I have the second order ODE \( G(x, \dot{x}, \ddot{x}) = 0 \), define \( y = \dot{x} \), and analyze the system:

\[ G(x,y,\dot{y}) \] (this is the equation for \( \dot{y} \) in implicit form) and \( \dot{x} = y \) (this is the equation for \( \dot{x} \)).

5. Derivation of Euler Eq.

A function is admissible if it is continuously differentiable on \([t_0, T]\) and satisfies

\[ x(t_0) = x_0, \quad x(T) = x_r \]

Suppose \( x'(t) \) is optimal. Let \( x(t) \) be another admissible soln. Define the deviation \( h(t) \)

\[ h(t) = x(t) - x'(t) \]
Admissibility ⇒ \[ x(t) = x^*(t) \quad \frac{dx}{dt} = x_T \Rightarrow h(T) = 0 \]
\[ x(0) = x^*(0) = x_0 \Rightarrow h(0) = 0 \]
and \( h(t) \) differentiable

Then \( x(t) = x^*(t) + ah(t) \) is also admissible. I rewrite the maximization problem in terms of the scalar \( a \).

\[
  g(a) = \int_{t_0}^{T} F(t, x(t), \dot{x}(t)) \, dt
  = \int_{t_0}^{T} \left( F\left(t, x^*(t) + a\dot{h}(t), x(t) + a\dot{x}(t) + a\dot{h}(t)\right) \right) \, dt
\]

Optimality of \( x^* \) implies that \( g(a) \) reaches a max at \( a = 0 \), which implies

\[
  \frac{dg}{da} \bigg|_{a = 0} = 0
\]

\[
  \frac{dg}{da} = \int_{t_0}^{T} \frac{dF}{da} \, dt =
\]

\[
  \int_{t_0}^{T} \left[ F\left(t, x^*, \dot{x}^*\right) h(t) + F\left(t, x^*, \dot{x}^*\right) \dot{h}(t) \right] \, dt = 0
\]

integrate by parts.

\[
  \int uv = vu - \int v \, du
\]

\[
  \int_{t_0}^{T} F_\dot{x} \, dh \, dt = \int_{t_0}^{T} F_\dot{x} \, dh =
\]

\[
  F_\dot{x} h(t) \bigg|_{t_0}^{T} - \int_{t_0}^{T} \left[ \frac{d}{dt} F_\dot{x}(t) \right] h(t) \, dt
\]
First term after last equality equals 0 by (1), implying

\[ (2') \quad \int_{t_0}^{T} \left[ F_x(x) - \frac{d}{dt} F_x(x) \right] h(t) dt = 0 \]

Notice that the term in brackets in the integrand is evaluated at \( a=0 \); consequently this term does not depend on the function \( h(t) \). I’ll abbreviate this term by writing it as \( z(t) \).

In order to show that (2’) implies the Euler equation, we need the Fundamental Lemma of calculus of variations:

If \( z(t) \) is continuous on \([t_0, T]\) and \( \int_{t_0}^{T} z(t) h(t) dt = 0 \) for all continuous \( h(t) \) then \( g(t) = 0 \).

Sketch of proof by contradiction. Suppose \( z(t) \neq 0 \) , e.g.

Find an interval over which \( z(t) \neq 0 \), let \( h(t) \neq 0 \) over only that interval. It is easy to see for this choice of \( h(t) \), the integral is not 0. This is our contradiction.

Apply fundamental lemma to (2’) to obtain Euler Equation

6. Special functional forms

1) Suppose \( F \) is linear-quadratic in \( x, \dot{x} \). Show Euler equation is linear ODE. Then show that this implies that \( \frac{dx}{dt} \) is a linear function of \( x \). Interpret this as linear control rule.
2) Suppose time enters only through discounting: \( F(t,x,\dot{x}) = e^{-rt}F(x,x) \). Show Euler equation is autonomous (no dependence on time.) In this case we have:

\[
\frac{d(F_x)}{dt} = \frac{d(e^{-rt} \tilde{F}_x)}{dt} = -re^{-rt} \tilde{F}_x \\
+ e^{-rt} \tilde{F}_{xx} \ddot{x} + e^{-rt} \tilde{F}_{xx} \dot{x} \\
= e^{-rt} \tilde{F}_x
\]

cancel \( e^{rt} \)

\[
\Rightarrow -r \tilde{F}_x + \tilde{F}_{xx} \ddot{x} + \tilde{F}_{xx} \dot{x} = \tilde{F}_x
\]

Another special case \( f(x) = \text{constant} \), see K&S pg 28

Other Important special cases

- \( F = F(t, \dot{x}) \) no dependence on \( x \) (in this case EE is 1st order ODE)
- \( F = F(x, \dot{x}) \) no dependence on \( t \)
- \( F = F(\dot{x}) \) no dependence on \( t \) or \( x \)
- \( F \) is linear in \( \dot{x} \) (talk about this later)

7. Necessary and sufficient conditions

We claimed that if \( x^* \) is optimal, then for all continuously differentiable \( h(t) \) with \( h(0) = h(T) = 0 \), \( a = 0 \) must maximize

\[
g(a) = \int_{t_0}^{T} F(t, x^*, ah, \dot{x}^* + ah) dt \\
g''(0) = \int_{t_0}^{T} \left[F_{xx} h^2 + 2F_{x\dot{x}} h \dot{h} + F_{\dot{x}\dot{x}} \dot{h}^2 \right] dt \leq 0
\]

(3)

A sufficient condition is that \( F \) is jointly concave in \( x, \dot{x} \). This condition often does not hold in interesting problems.

The Legendre Condition, \( F_{xx}(t, x^*, \dot{x}^*) \leq 0 \) is necessary (It isn't sufficient).
Show necessity of Legendre condition

\[ 2F_{xx} h\dot{h} = F_{xx} \frac{d(h^2)}{dt} \Rightarrow \]
\[ 2 \int_{t_0}^{T} F_{xx} h\dot{h} = \int_{t_0}^{T} F_{xx} \frac{d(h^2)}{dt} dt = \]
\[ F_{xx} h^2 \bigg|_{t_0}^{T} - \int_{t_0}^{T} h^2 \frac{d}{dt}(F_{xx}) dt \]

\[ = 0 \]

So (3) can be rewritten

\[ g''(0) = \int_{t_0}^{T} \left\{ F_{xx} - \frac{d}{dt} \left( F_{xx} \right) \right\} h^2 + F_{xx} (\dot{h})^2 dt = \int_{t_0}^{T} (Q h^2 + P \dot{h}^2) dt \]

\[ = Q, \quad \Rightarrow = P_t \]

Lemma: Given continuous functions \( P(t), Q(t) \), a necessary condition for \( \int_{t_0}^{T} \left[ Q_t h^2 + P_t \dot{h}^2 \right] dt \leq 0 \) is \( P(t) \leq 0 \). (This inequality is just the Legendre condition.)

Geometric argument. Suppose \( P > 0 \) over some interval.
Choose \( h(t) \) so that \( |h(t)| \) arbitrarily small, but \( |\dot{h}(t)| \) arbitrarily large. With this choice, the term involving \( P \) is positive and non-negligible, and the term involving \( Q \) is negligible, so that the integral is positive. This is our contradiction.

8. Different boundary conditions

- \( T \) fixed, \( x(t) \) free (free end value)
- \( T \) free, \( x(T) \) fixed
- both \( x(t) \) and \( T \) free
- There is a scrap function \( S(x(t),T) \) and either \( x(T) \)
- and/or \( T \) is free

\( x(0) \) may be free (jump states)
\( t_0 \) may be free (time you begin extracting from mine)

See pg 60-61 K&S for boundary conditions

Argument for free terminal value of state, \( x(T) \) is free:

\[
g(a) = \int_{t_0}^{T} F(t, x^* + ah, \dot{x}^* + ah) dt \quad \text{must be maximized at } a = 0 \Rightarrow g(0)=0
\]
\[ g'(0) = \int_{t_0}^{T} \left( F_x h + F_x \dot{h} \right) dt \]

The second term is
\[ \int_{t_0}^{T} F_x \dot{h} \left| \right. _{t_0}^{T} - \int_{t_0}^{T} h \frac{dF_x}{dt} dt \]

We get same stuff as before, but now \( h(T) \) need not equal 0, so we have the extra condition
\[ F_x \left( T, x \dot{(T)} \dot{x}(t) \right) = 0 \]

This condition is called a transversality condition (TC).

Interpretation (fish story): If the terminal stock is a choice variable (i.e., \( x(T) \) is free) marginal utility of consuming final unit should be 0 at \( T \), since there is no reason to keep stock.

When we "lose" a B.C. it is replaced by a transversality condition.

If the final time, \( T \) is free (i.e. a choice variable), transversality condition is
\[ (4) \quad F \left( T, x \dot{(t)} \dot{x}(T) \right) - \dot{x} F_x(T...) = 0 \]

Interpretation (fish story): If I extended the time of consumption by one unit, I would get an extra \( F \) units of utility. When \( T \) is chosen optimally, this extra benefit should equal the loss incurred by changing harvest by the amount needed to hold constant the terminal value of the state - the second term of (4). (If \( x(T) \) is fixed, then by definition I have to hold its terminal value constant. If \( x(T) \) is chosen optimally, there is no benefit from changing that value -- I want to hold its terminal value fixed.)

9 An example: extraction of a nonrenewable resource

Extraction of a nonrenewable resource. Example: Rate of extraction is \( -\dot{x} = y \) with price \( p(y) \), utility of consumption \( U(y) \)
\[ U'(y) = p(y), \quad U''(y) = p'(y) < 0 \]
Social planner's problem is

\[
\max \int_0^T e^{-\gamma t} U(y) dt \quad \text{subject to } x_0 \text{ given, } x_T = 0, \ T \text{ free}
\]

Euler equation

\[
F_x = 0 = -\frac{d}{dt}[e^{-\gamma t}U'(y)]
\]

\[\Rightarrow e^{-\gamma t}U'(y) = c, \ a \text{ constant}\]

This is Hotelling Rule: PDV of Marginal Utility (= price) constant, so price growing at rate of interest.

(Note that whenever \(F_x = 0\), EE is a first order ODE, much easier to solve than 2nd order ODE.)

Legendre condition satisfied: \(F_{\dot{x}} = -U'(-\dot{x}), \ F_{\ddot{x}} = U''(-\dot{x}) < 0\).

Transversality condition: \(\int_0^T e^{-\gamma t} [U(-\dot{x}) + \dot{x}U'(-\dot{x})] = 0 \quad (4')\)

Transversality condition provides a boundary condition for the ODE given to us from the Euler Equation. This boundary condition, implied by equation (4') is \(\dot{x} = 0\). This equality is, \(\dot{x}=0\), is obviously sufficient for TC. Show \(\dot{x}=0\) is necessary for TC. First consider case where \(T\) is finite (as is the case for linear example below).

Suppose \(\dot{x}(T) \neq 0\), TC \(\Rightarrow\)

\[
\left[\frac{U(-\dot{x})}{\dot{x}} + U'(-\dot{x})\right] = 0
\]

This equation says that average and marginal utilities are equal at time \(T\). However, since \(U\) is concave in \(-\dot{x}\), average utility is always greater than marginal utility. Thus, the hypothesis that \(\dot{x}(T) \neq 0\) leads to a contradiction, so in this case (\(T\) is finite) they hypothesis is false.

Now consider the possibility that \(T\) is infinite (as will be the case if \(\lim_{y \to 0} U'(y) = \infty\)) In this case \(\dot{x}\) must approach 0 as \(T\) becomes large. (Extraction cannot remain bounded away from 0 forever because...
the initial stock of the nonrenewable resource is finite.

In order to see how you would actually use the transversality condition to solve a control problem, consider the previous problem, but specialize the general utility function to the quadratic case. In other words, assume linear marginal utility:

\[ U'(y) = (a - by) = a + b\dot{x} \]

The EE becomes:

\[ a + b\dot{x} = ce^{rt} \]

(price grows at rate \( r \))

\[ x(0) \text{ given, } x(T) = 0 \]

integrate Euler equation, use B.C.

\[
\begin{align*}
 x_0 - 0 &= \int_0^T -\dot{x} dt = \int_0^T \frac{(a - ce^{rt})}{b} dt \\
 x_0 &= \int_0^T \frac{(a - ce^{rt})}{b} dt = \frac{aT}{b} - \frac{c}{br} (e^{rT} - 1)
\end{align*}
\]

This is one equation in 2 unknowns, \( c \), and \( T \).

Use \( x(T) = 0 \) in \( a + b\dot{x} = ce^{rT} \) \( \Rightarrow a = ce^{rT} \)

As an exercise, solve problem above for monopolist rather than social planner.