Partial solution to Problem set 9

Answer to part a)

$y = F + \gamma N$

The DPE for the autonomous problem is:

$$rJ(N) = \max_F \left(ay - \frac{b}{2}y^2 + J'(N)(AN + BF) \right)$$

I am going to "guess" that the value function is quadratic in the state:

$$J = \lambda + \mu N + \nu \frac{N^2}{2}$$

Substitute this guess into the DPE to obtain

$$r\left(\lambda + \mu N + \nu \frac{N^2}{2}\right) = \max_F \left(ay - \frac{b}{2}y^2 + (\mu + \nu N)(AN + BF) \right)$$

Now I'm going to ask Scientific Workplace to find the max

$ay - \frac{b}{2}y^2 + (\mu + \nu N)(AN + BF)$ Candidate(s) for extrema:

$$\frac{1}{2b}\left(PN^2 + KN + L\right)$$

$$\begin{aligned} P &\equiv B^2\nu^2 + 2\nu Ab - 2\nu Bb\gamma \\ K &\equiv 2aB\nu + 2B^2\mu\nu + 2\mu Ab - 2\mu Bb\gamma \\ L &\equiv a^2 + 2aB\mu + B^2\mu^2 \end{aligned}$$

The optimal control rule is $F = \frac{a - b\gamma N + B\mu + B\nu N}{b}$

The maximized DPE is

$$r\left(\lambda + \mu N + \nu \frac{N^2}{2}\right) = \frac{1}{2b}\left(PN^2 + KN + L\right)$$

Equating coefficients in N^2 implies

$$r\nu = \frac{P}{b}$$

or

$$r\nu = \frac{B^2\nu^2 + 2\nu Ab - 2\nu Bb\gamma}{b}$$

or

$$\frac{1}{b}\left(B^2\nu^2 + 2\nu Ab - 2\nu Bb\gamma\right) - \nu = 0,$$

Solution is: $\{\nu = 0\}, \left\{\nu = b\frac{-2A+2B\gamma+1}{B^2}\right\}$

By the parameter assumption, $b\frac{-2A+2B\gamma+1}{B^2} > 0$. Therefore, the stable root is $\nu = 0$, as was to be shown.

Sketch of answer to part b (Here I am letting the parameters of the value function be ρ_0 and ρ).

I start out with the autonomous DPE and the "guess" (which I confirmed for the scalar case above) that the value function is linear in the state. (In a sense, part (a) of this problem is redundant. If you can show that the linear value function satisfies the DPE, then because the solution is unique, it is not necessary to show that the quadratic part must be 0. However, it is good practice – and fun!)

DPE:

$$r\left(\rho_0 + \rho' N\right) = \max_F \left(ay - \frac{b}{2}y^2 - cF + \rho'\left(AN + BF\right)\right). \qquad (1)$$

The first order condition and the definition of y imply equation (1), where α is given by equation (2) of the problem set. Substituting (1) into the definition of y implies that at the optimum

$$y^* = \alpha.$$

Substituting the optimal rule into equation (1) of the answer key implies

$$r\left(\rho_0 + \rho' N\right) = \left(a\alpha - \frac{b}{2}\alpha^2 - c\left(\alpha - \gamma' N\right) + \rho'\left(AN + B\left(\alpha - \gamma' N\right)\right)\right).$$

Equating coefficients of N implies

$$r\rho = c\gamma + \left(A' - \gamma B'\right)\rho.$$

Solving for ρ implies equation (3) of the problem set.