Solutions to Problem Set 4

ARE 261

September 30, 2002

Question 1

The control is u_i, the state is y_i and the co-state variable is λ_i. The first order conditions with respect to each of these variables are given by

$$\frac{\partial L}{\partial u_i} = \left(\frac{\partial F}{\partial u_i} + \lambda_i\right)\Delta x = 0, \qquad i = 0, 1, ...n-1 \tag{1}$$

$$\frac{\partial L}{\partial y_i} = \begin{cases} \frac{\partial F}{\partial y_i}\Delta x + \frac{(\lambda_i - \lambda_{i-1})}{\Delta x} = 0, & i = 1, ...n-1 \\ -\frac{\lambda_{n-1}}{\Delta x} = 0, & i = n \end{cases} \tag{2}$$

$$\frac{\partial L}{\partial \lambda_i} = (y_{i+1} - y_i) - u_i \Delta x = 0 \qquad i = 0, 1, ...n-1 \tag{3}$$

The first and the third equation hold for $i = 0, ..., n-1$ while the second equation is different for y_i $(i = 1, 2..., n-1)$ and and y_n. (Since y_0 is given there is no first order condition for this variable.)

Question 2

The Lagrangian can be written as

$$L = \sum_{i=0}^{n-1} \left(H_i \Delta x - \lambda_i \left(y_{i+1} - y_i\right)\right) \tag{4}$$

The first order conditions in terms of the Hamiltonian are

$$\frac{\partial L}{\partial u_i} = \frac{\partial H_i}{\partial u_i}\Delta x = 0 \implies \frac{\partial H_i}{\partial u_i} = 0 \quad i = 0, 1...n-1 \tag{5}$$

$$\begin{array}{c} \frac{\partial L}{\partial y_i} = \left(\frac{\partial H}{\partial y_i}\Delta x + ((\lambda_i - \lambda_{i-1}))\right) = 0 \quad i = 1, 2...n-1 \implies \\ -\frac{\partial H}{\partial y_i} = \frac{(\lambda_i - \lambda_{i-1})}{\Delta x} \end{array} \tag{6}$$

$$\frac{\partial L}{\partial y_n} = -\lambda_{n-1} = 0 \tag{7}$$

$$\begin{array}{c} \frac{\partial L}{\partial \lambda_i} = \left(-(y_{i+1} - y_i) + \frac{\partial H}{\partial \lambda_i}\Delta x\right) = 0 \implies \\ \frac{y_{i+1} - y_i}{\Delta x} = \frac{\partial H}{\partial \lambda_i} = u_i \end{array} \tag{8}$$

Equation (7) implies that $\lambda_{n-1} = 0$ when y_n is free.

Question 3

When there is a scrap function the Langrangian is

$$L^* \equiv L + f(y_n)$$

where L is given by equation (19). Equation (7) is replaced by

then the derivative of the Lagrangian with respect to y_n changes to

$$\frac{\partial L^*}{\partial y_n} = \frac{\partial f}{\partial y_n} - \lambda_{n-1} = 0 \tag{9}$$

Hence

$$\lambda_{n-1} = \frac{\partial f}{\partial y_n} \tag{10}$$

Question 4

If I want to change the length of each stage, Δx, and keep the length of the problem constant, I need to define the length of the problem as $(n-1)\Delta x \equiv T$, so $n = \frac{T+\Delta x}{\Delta x}$. As $\Delta x \to 0$, $n \to \infty$. (Each period becomes short, and there are many periods.) The summation in the equation (1) becomes an integral and the difference equations become differential equations. Formally taking the limit of the necessary conditons gives

$$\frac{\partial H}{\partial u_i} = 0 \tag{11}$$

$$\frac{\partial H}{\partial y_i} = -\dot{\lambda} \tag{12}$$

$$\frac{\partial H}{\partial \lambda_i} = \dot{y} \tag{13}$$

Note that in the discrete time problem, the Hamiltonian and the necessary conditions are defined for $i = 0, 1, 2....$ In the continuous time problem, the Hamiltonian is defined at every point in time between 0 and T, and the necessary conditions must hold at every point. (It is as if there were "uncountably many first order conditions" in the continuous limit.)

Question 5

The current value Hamiltonian for the control problem is given by

$$H = u - \frac{u^2}{2} - \frac{x^2}{2} + \lambda(u - bx) \tag{14}$$

And the necessary conditions are

$$\frac{\partial H}{\partial u} = 1 - u + \lambda = 0 \tag{15}$$

$$\frac{\partial^2 H}{\partial u^2} = -1 < 0 \tag{16}$$

$$\dot{\lambda} = r\lambda - \frac{\partial H}{\partial x} = r\lambda + x + b\lambda \tag{17}$$

$$\dot{x} = \frac{\partial H}{\partial \lambda} = u - bx \tag{18}$$

The terminal conditions are replaced by the steady state conditions.

Next we want to write a system of differential equations in the state and the control which means that we have to get rid of λ from the necessary conditions. But first differentiate the first necessary condition with respect to time. This gives

$$\dot{u} = \dot{\lambda} \tag{19}$$

Using this along with the other first order conditions gives the following two differential equations in the state and control

$$\dot{u} = (r + b)(u - 1) + x \tag{20}$$

$$\dot{x} = u - bx \tag{21}$$

To find the steady states (which substitute for the terminal conditions) we set the two differential equations equal to zero. This imples that the steady state values of u^* and x^* are given by

$$u^* = \frac{b(r + b)}{1 + b(r + b)} \tag{22}$$

$$x^* = \frac{r + b}{1 + b(r + b)} \tag{23}$$

The two isoclines we are interested in, namely $\dot{u} = 0$ and $\dot{x} = 0$ are given by

$$u = 1 - \frac{x}{r+b} \quad (\dot{u} = 0) \tag{24}$$

$$u = bx \quad (\dot{x} = 0) \tag{25}$$

To determine the directional arrows for the $\dot{u} = 0$ isocline differtiate the equation for the isocline with respect to x. This gives

$$\frac{d\dot{u}}{dx} = 1 \tag{26}$$

This implies that to the right of the isocline u increases and to the left it decreases. Similarly to determine the directional arrows for the $\dot{x} = 0$ isocline we differntiate the equation for the isocline which gives

$$\frac{d\dot{x}}{du} = 1 \tag{27}$$

So above the isocline x increases and decreases below the isocline.

We determine the stability of the steady state by looking at the A matrix which for this problem is given by

$$\Big(A\Big) = \begin{pmatrix} r+b & 1 \\ 1 & -b \end{pmatrix}$$

The determinant of matrix A is negative which implies that the steady state is a saddle point.

Finally, we need to determine the optimal control rule. For a linear quadratic system with one state the optimal control rule is given by the converging separatrix. We know that the separatrix is a line whose slope is equal to the slope of the trajectory. The slope of the optimal trajectory is given by

$$\frac{du}{dx} = \frac{\dot{u}}{\dot{x}} = \frac{(r+b)(u-1)+x}{u-bx} \tag{28}$$

Let $u = k_1 + k_2 x$ denote the separatrix. We know that $\frac{du}{dx} = k_2$. Substituting for u from the equation of the separatrix and equating the coefficients of x gives the following two equations for k_1 and k_2

$$k_1 k_2 = (r+b)(k_1 - 1) \tag{29}$$

$$k_2^2 - bk_2 - 1 - (r+b)k_2 = 0 \tag{30}$$

These in turn imply that $k_2 = \frac{\alpha+\sqrt{\alpha^2+4}}{2}$ or $k_2 = \frac{\alpha-\sqrt{\alpha^2+4}}{2}$ and $k_1 = \frac{r+b}{r+b-k_2}$, where $\alpha = r+2b$. Since the optimal path is given by the converging separatrix we can show that the correct root is $k_2 = \frac{\alpha - \sqrt{\alpha^2+4}}{2}$. With this root the optimally controlled system converges to the steady state. The other root gives you the formulae for the diverging seperatrix – a path that diverges from the steady state. Therefore, the optimal control rule is

$$u = \frac{r+b}{r+b-k_2} + k_2 x \tag{31}$$

where $k_2 = \frac{\alpha - \sqrt{\alpha^2+4}}{2}$.