Solutions to Problem Set 8

ARE 261

December 15, 2004

Question 1

1. The dynamic programming equation (DPE) for the control problem is

$$-J_t(x_t,t) = \max_{u_t}\left\{-\frac{1}{2}e^{-rt}(u_t^2+x_t^2)+J_x(x_t,t)(x_t+u_t)\right\} \tag{1}$$

   where $J(x_t,t)$ is the value function, and $J_t(x_t,t)$ and $J_x(x_t,t)$ are partial derivatives of the value function.

2. $J(x_T,T) = -e^{-rT}\frac{ax_T^2}{2}$.

3. Substitute the guess for the value function, that is $J(x_t,t)=s(t)\frac{x_t^2}{2}$, into the DPE, differentiate with respect to u_t, and set the expression to zero. Note that $J_t(x_t,t) = \dot{s}\frac{x_t^2}{2}$ and $J_x(x_t,t) = sx$. This gives the following policy function

$$u_t = e^{rt}s(t)x_t \tag{2}$$

   Substitute the optimal policy function into the DPE. This in turns gives the differential equation that solves $s(t)$

$$\dot{s} = -e^{rt}s^2 + e^{-rt} - 2s_t \tag{3}$$

   With the boundary condition that $s(T) = -ae^{-rT}$. It is easier to work with a transformation of $s(t)$. Let $z(t) = e^{rt}s(t)$. Then

$$\frac{dz}{dt} = -z\left(t\right)^2 - \left(2-r\right)z\left(t\right)+1 \tag{4}$$

   With the boundary condition that $z_T = -a$.

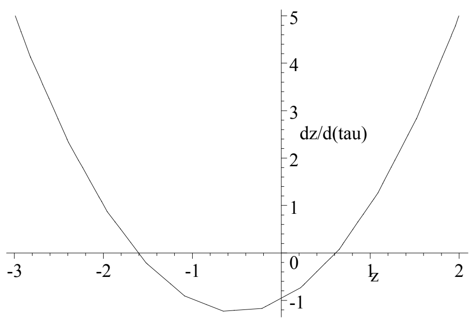
4. λ_t from the Maximum Principle is equal to $J_x(x_t,t)$ from the DPE.

5. We already have an autonomous differential equation in $z(t)$. By solving this equation "backwards" we convert the boundary condition at the terminal period into a boundary condition at the initial time. Redefine

the differential equation for $z(t)$ in terms of τ where $\tau = T - t$, the time remaining. In terms of τ the differential equation is

$$\frac{dz}{d\tau} = z(\tau)^2 + (2 - r)\, z(\tau) - 1 \tag{5}$$

This equation has two steady states, one positive and one negative. These are solved for by setting $\frac{dz}{d\tau} = 0$ and then solving the quadratic in z. The positive root is equal to $z_1 = \frac{-(2-r)+\sqrt{(2-r)^2+4}}{2}$ and the negative root is equal to $z_2 = \frac{-(2-r)-\sqrt{(2-r)^2+4}}{2}$. The negative root is stable while the positive root is unstable. The figure shows the graph of $\frac{dz}{d\tau} = 0$ for $r = 1$

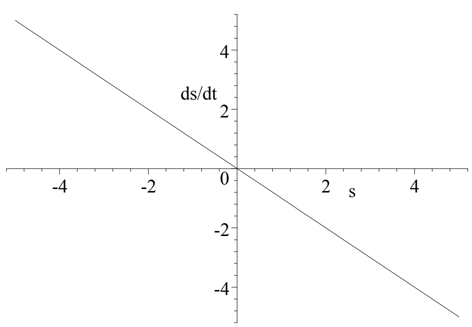


In the steady state, we have

$$s^* = z_2 e^{-rt},$$

so

$$\frac{ds}{dt} = -r z_2 e^{-rt} = -rs$$



6. So long as the initial point $-a$ lies to the left of the positive root z_1, the differential equation for s will converge to the steady state as $T \to \infty$. This implies that if $a > -z_1$ then the differential equation will converge.

7. By part 3, equation (2), the optimal control rule is

$$\begin{aligned} u_t &= e^{rt} s(t) x_t \\ &= z_2 x_{t.} \end{aligned}$$

with $z_2 = \frac{-(2-r)-\sqrt{(2-r)^2+4}}{2}$.

The derivative is:

$$\frac{dz_2}{dr} = \frac{1}{2} + \frac{1}{2}\left[(2-r)^2+4\right]^{-1/2}(2-r) > 0$$

Confirm that this expression is positive by noting:

$$\begin{aligned} \left((2-r)^2+4\right) &> (2-r)^2 \Rightarrow \\ \left((2-r)^2+4\right)^{1/2} &> -(2-r) \Rightarrow \\ 1 &> -\left((2-r)^2+4\right)^{-1/2}(2-r) \Rightarrow \\ 1+\left((2-r)^2+4\right)^{-1/2}(2-r) &> 0 \end{aligned}$$

Since $u = z_2 x$, conclude that

$$\text{sign}\,\frac{\partial u}{\partial r} = \text{sign}\ x.$$

Question 2

Let $J(1)$ denote the value when the state is bad and $J(0)$ the value when the state is good. From the set up its easy to see that

$$x^* = 0 \qquad (bad \quad state)$$

$$J(1) = \frac{D_b}{1-\beta} \tag{6}$$

$$J(0) = \min_{x_t}\{c(x_t) + \beta(1-p(x_t))J(0) + \beta p(x_t)J(1)\} \tag{7}$$

Derive the first order condition for the second DPE. This gives

$$\frac{dc(x_t)}{dx_t} + \beta\frac{dp(x_t)}{dx_t}\left[J(1) - J(0)\right] = 0 \tag{8}$$

Let $f(x, J(0), J(1), \beta)$ denote the left hand side of the first order condition. Totally differentiating f and then rearranging the terms gives

$$\frac{df}{d\beta} = f_x\frac{dx}{d\beta} + f_\beta + f_{J(0)}\frac{dJ(0)}{d\beta} + f_{J(1)}\frac{dJ(1)}{d\beta} = 0 \tag{9}$$

From the first order condition

$$f_x = \frac{d^2c(x)}{dx^2} + \beta(J(1) - J(0))\frac{d^2p(x)}{dx^2} \tag{10}$$

$$f_\beta = \frac{dp(x)}{dx}(J(1) - J(0)) \tag{11}$$

$$f_{J(0)} = -\beta\frac{dp(x)}{dx} \tag{12}$$

$$f_{J(1)} = \beta\frac{dp(x)}{dx} \tag{13}$$

Note that $f_x \geq 0$ from the second order condition. We still need to compute the differential of the value functions with respect to β. From the expression for $J(1)$ it is easy to see that

$$\frac{\partial J(1)}{\partial \beta} = \frac{D_b}{(1-\beta)^2} \geq 0 \tag{14}$$

For $J(0)$ we use the dynamic envelope theorem on the DPE (7) after we have substituted for the optimal policy function. This implies that

$$\begin{aligned} \frac{\partial J(0)}{\partial \beta} &= (1-p(x))J(0) + p(x)J(1) \\ &+\beta\left[p(x)\frac{D_b}{(1-\beta)^2} + (1-p(x))\frac{\partial J(0)}{\partial \beta}\right]. \end{aligned}$$

It gives

$$\frac{\partial J(0)}{\partial \beta} = \frac{(1-p)J(0) + \frac{p}{1-\beta}J(1)}{1-\beta(1-p)} > 0. \tag{15}$$

After making all the substitutions into (9) and after a lot of simplification we get

$$\frac{dx}{d\beta} = -\frac{\frac{dp(x)}{dx}(J(1)-J(0))}{[1-\beta(1-p)]\, f_x} \tag{16}$$

To see the sign of those terms in $\frac{dx}{d\beta}$:

$$\begin{aligned} f_x = \frac{d^2c(x)}{dx^2} + \beta(J(1)-J(0))\frac{d^2p(x)}{dx^2} &> 0, \qquad (by \quad S.O.C.) \\ \frac{dp(x)}{dx}(J(1)-J(0)) = -\frac{1}{\beta}\frac{dc(x_t)}{dx_t} &< 0, \qquad (by \quad F.O.C.) \end{aligned}$$

Consequently,

$$\frac{dx}{d\beta} > 0.$$

As β increases the agent chooses to increase x. Higher discount factor increases the optimal abatement under a good state.

Question 3

1. The farmer wants to pick an action, x, to maximize the sum of profits subject to the equation of motion for soil quality.

$$\max_{\{x_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \pi(S_t, x_t) \tag{17}$$

subject to

$$S_{t+1} = f(S_t, x_t) \tag{18}$$

2. The corresponding DPE is

$$V(S_t) = \max_{x_t} \pi(S_t, x_t) + \beta V(S_{t+1}) \tag{19}$$

where V denotes the value function.

3. The following steps are involved in solving for the Euler equation.

- Solve for the first order condition
- Solve for the envelope condition
- Advance both the first order and the envelope condition one period
- Eliminate the Value function from the advanced envelope condition

We begin by differentiating the DPE with respect to x_t and setting the expression to zero. This gives

$$V'(S_{t+1}) = -\frac{\frac{\partial \pi(S_t, x_t)}{\partial x_t}}{\beta \frac{\partial f(S_t, x_t)}{\partial x_t}} \tag{20}$$

The envelope condition (using the envelope theorem and treating the state as a parameter) is

$$V'(S_t) = \frac{\partial \pi(S_t, x_t)}{\partial S_t} + \beta V'(S_{t+1}) \frac{\partial f(S_t, x_t)}{\partial S_t} \tag{21}$$

Next advance the first order condition and the envelope condition one period. This gives,

$$V'(S_{t+2}) = -\frac{\frac{\partial \pi(S_{t+1}, x_{t+1})}{\partial x_{t+1}}}{\beta \frac{\partial f(S_{t+1}, x_{t+1})}{\partial x_{t+1}}} \tag{22}$$

$$V'(S_{t+1}) = \frac{\partial \pi(S_{t+1}, x_{t+1})}{\partial S_{t+1}} + \beta V'(S_{t+2}) \frac{\partial f(S_{t+1}, x_{t+1})}{\partial S_{t+1}} \tag{23}$$

Finally we use the equations we have to eliminate value function from the advanced envelope condition. This gives the Euler equation as

$$-\frac{\frac{\partial \pi(S_t,x_t)}{\partial x_t}}{\beta \frac{\partial f(S_t,x_t)}{\partial x_t}} = \frac{\partial \pi(S_{t+1},x_{t+1})}{\partial S_{t+1}} - \frac{\frac{\partial \pi(S_{t+1},x_{t+1})}{\partial x_{t+1}}}{\frac{\partial f(S_{t+1},x_{t+1})}{\partial x_{t+1}}} \frac{\partial f(S_{t+1},x_{t+1})}{\partial S_{t+1}} \tag{24}$$

or

$$-\frac{\pi_x(S_t,x_t)}{f_x(S_t,x_t)} = \beta \left\{ \pi_s(S_{t+1},x_{t+1}) - \frac{\pi_x(S_{t+1},x_{t+1})}{f_x(S_{t+1},x_{t+1})} f_s(S_{t+1},x_{t+1}) \right\}$$

4. At the steady state $S_t = S_{t+1} = S$ and $x_t = x_{t+1} = x$. These values are solved for by solving the equation of motion and the Euler equation at the steady state. These are

$$S = f(S,x) \tag{25}$$

$$-\frac{\pi_x(S,x)}{f_x(S,x)} = \beta \left\{ \pi_s(S,x) - \frac{\pi_x(S,x)}{f_x(S,x)} f_s(S,x) \right\} \tag{26}$$

Question 4

1. The optimal control problem is

$$\max_{\{h_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(h_t) \tag{27}$$

subject to

$$S_{t+1} = (S_t - h_t)^{\alpha} \tag{28}$$

2. The corresponding DPE, where V denotes the value function, is

$$V(S_t) = \max_{h_t} \log(h_t) + \beta V(S_{t+1}) \tag{29}$$

while the boundary condition is that $h_T = S_T$ which implies that $V(S_T) = \log(S_T)$.

3. An inductive proof involves three steps:

- Show that the guessed form is true for $\tau = 0$
- Assume that the guessed form is true for $\tau = s$
- Show that the guessed form is true for $\tau = s + 1$

where $\tau = T - t$, the time to go and the guess is that $V(S_\tau) = A_{0\tau} + A_\tau \log(S_\tau)$. Now $\tau = 0$ when $t = T$. Consequently, $V(S_{\tau=0}) = V(S_T) = \log(S_T)$. So when $\tau = 0$ the value function has the guessed form with $A_{00} = 0$ and $A_0 = 1$. Now assume that the guessed form is true for $\tau = s$. This implies that $V(S_s) = A_{0s} + A_s \log(S_s)$. Finally we need to show that the guessed form is true for $\tau = s+1$. Write the DPE going forward but in terms of s and $s+1$ (note that in terms of t, $s+1 < s$). This gives

$$V(S_{s+1}) = \max_{h_{s+1}} \quad \log(h_{s+1}) + \beta V(S_s) \tag{30}$$

Substitute in for $V(S_s)$ and $S_s = (S_{s+1} - h_{s+1})^\alpha$ and then derive the first order condition. This gives

$$\frac{1}{h_{s+1}} - \frac{\alpha\beta A_s}{S_{s+1} - h_{s+1}} = 0 \tag{31}$$

Which in turn implies that $h_{s+1} = \frac{S_{s+1}}{1+\alpha\beta A_s}$. Substitute this back into the DPE. This gives

$$\begin{aligned} V(S_{s+1}) \quad = \quad & \log(S_{s+1}) - \log(1+\alpha\beta A_s) + \beta A_{0s} + \\ & \beta\alpha A_s \left[\log(S_{s+1}) + \log(\alpha\beta A_s) - \log(1+\alpha\beta A_s)\right] \end{aligned} \tag{32}$$

This has the guessed functional form so long as

$$\begin{aligned} A_{0s+1} &= \beta A_{0s} + \beta\alpha A_s \log(\alpha\beta A_s) - (1+\alpha\beta A_s)\log(1+\alpha\beta A_s) \\ A_{s+1} &= 1 + \alpha\beta A_s \end{aligned} \tag{33}$$

Then $V(S_{s+1}) = A_{0s+1} + A_{s+1}\log(S_{s+1})$.

4. The last two equations are the difference equations for $A_{0\tau}$ and A_τ. The control rule is given by

$$h_{\tau+1} = \frac{S_{\tau+1}}{1+\alpha\beta A_\tau} \tag{34}$$

The change in the fraction of the stock harvested depends on the change in A_τ. From the difference equation we know that $A_{\tau+1} > A_\tau$. This implies that as the time remaining increases the agent harvests a larger share of the stock. Alternatively, the share harvested decreases as $t \to T$. To ensure that the steady state of the difference equation for A_τ (33) is stable, it must be true that the difference equation cuts the 45^o line from above. This requires that $\alpha\beta < 1$. Assuming non-negative harvesting in the steady state, the growth equation (28) requires $\alpha \geq 0$.

5. The steady state control rule for the infinite horizon is $h = \frac{S}{1+\alpha\beta A}$. This is the same as the rule we would have obtained if we had started from the infinite horizon.

Question 5

The lobby groups wants to choose c_t in every period to maximize its expected profits. This amounts to

$$\max_{c_t} \sum_{t=0}^{\infty} \beta^t \left[E(\text{rent}) - c_t\right] \tag{35}$$

The DPE when the friendly party is in office is given by

$$J(F) = \max_{c_t} \left\{b - c_t + \beta \left[p(c_t)J(F) + (1 - p(c_t))J(U)\right]\right\} \tag{36}$$

while the DPE when the unfriendly party is in office is

$$J(U) = \max_{c_t} \left\{-c_t + \beta \left[(1 - q(c_t))J(F) + q(c_t)J(U)\right]\right\} \tag{37}$$