Problem set 9
(Inspired by a paper by Craig Bond and Hossein Farzin)

Consider the following linear-quadratic control problem: N is a $n-$dimensional vector of stocks of nutrients and F is the flow of fertilizer application. The flow of nutrients available to a plant is $y = F + \gamma' N$, and the output (revenue) is $\pi = ay - \frac{b}{2}y^2$. The unit cost of nutrients in cF and the equation of motion for nutrients is

$$\dot{N} = AN + BF.$$

Note that by including the constant 1 in the vector N, this equation of motion includes a constant term. The constant discount rate is r. The farmer wants to maximize the present value of the flow of profits over an infinite horizon. Suppose that F is unconstrained.

Assume that $b > 0$ (so that profits are concave in nutrients, $B > 0$ (so that an increase in fertilizer increases the stock of nutrients) and $A < 0$ (so that the stock of nutrients decays to a steady state, in the absence of additional inputs by way of fertilizer. The steady state is equal to the constant in the equation of motion.)

a)Write down the continuous time dynamic programming equation. Guess that the value function is linear-quadratic in the state. For the case where N is a scalar, show that the stable (i.e. the non-positive) root of the algebraic Riccati equation is 0. Conclude that for this problem, the value function is linear (rather than quadratic) in the state. Provide an intuitive explanation.

b) For the case where N is a vector, show that the value function $J(N) = \rho_0 + \rho' N$ solves the Bellman equation, and that the optimal level of fertilization is

$$F = \alpha - \gamma' N \tag{1}$$

where

$$\alpha = \frac{a - c + B'\rho}{b} \tag{2}$$

$$\rho = c\left(rI - A' + \gamma B'\right)^{-1} \gamma. \tag{3}$$

Note that for this problem we can obtain a closed form solution by solving the linear equation (3). We can interpret α as the target level of nutrients, so $\lambda \equiv c(\alpha - \gamma' N)$ is the cost of reaching this target. That is, λ is an index of the value of the stock, and it can be written in closed form.

c) Show that a sufficient condition for this solution to satisfy the transversality condition

$$\lim_{t \to \infty} e^{-rt} J_N N$$

is that N is bounded. Note that N converges to a globally asymptotically stable (GAS) steady state (and thus the transversality condition is satisfied) provided that the real parts of the eigenvalues of $A - B\gamma'$ are negative. (This is a sufficient but not necessary condition for the transversality condition.)

d) Sketch the dynamics for the two cases where N is a scalar and where it has two elements.

e) How does the problem change if you impose a constraint of the form $g \leq F \leq h$, where g and h are constants?