Problem Set 8

ARE 261

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1) The control problem is

$$\begin{aligned} &\max -\frac{1}{2}\int_o^T e^{-rt}(u^2+x^2)dt - e^{-rT}\frac{ax^2}{2} \\ s.t.\ \dot{x} &= x+u, \quad x_0 \text{ given.} \end{aligned}$$

where a is a parameter.

a)Write down the dynamic programing equation.

b)Write the boundary condition for this equation; that is, what is the value function at t = T?

c) "Guess" that the value function is quadratic: $J(x,t) = s(t)x^2/2$.

Find the differential equation that $s(t)$ solves, and the boundary condition for this equation. (This is called a Ricatti differential equation.)

At this point it is useful to recognize that you can simplify the Ricatti differential equation by using a transformation of s. Define $z(t) = e^{rt}s(t)$. Using this definition and your previous equation for $\dot{s}$, find the equation for $\dot{z}$ and the boundary condition for z.

It is convenient to work with z rather than s because the differential equation for z does not depend on calendar time, t – that is, the equation for z is autonomous.

d) Remember that you previously solved (essentially) this problem using the Maximum Principle. There you "guessed" that the costate variable was linear in the state. What is the relation between these two approaches?

e) Define $\tau = T - t$, the "time to go" (until the end of the horizon) We want to consider the limiting form of the original problem as $T \to \infty$. From the definition of τ we have $d\tau = -dt$. By working with τ ("time to go") rather than with t (the calendar time) we have "reversed the clock". Note that as $T \to \infty$, $\tau \to \infty$ for any finite t (finite calendar time). Use your previous equation for $\frac{dz}{dt}$ to find the equation for $\frac{dz}{d\tau}$. Graph this equation as a function of z. Notice that there are two steady states. Which of these is stable in the reversed system (by "reversed system" I mean that the independent variable is τ rather than t)? The algebraic (as opposed to differential) equation for the steady stae value of z is known as the algebraic Ricatti equation.

f) For what set of values of the parameter a does s converge to a steady steady state as $T \to \infty$?

g) Find the how the optimal control rule changes as r changes.

2) Do exercise 2 on page 5.2 of the notes. (Complete the sketch of the solution that I provided in class.)

3) A farmer wants to maximize the present discounted value of profits, with a discount factor β. Profits in a period, $\pi(S, x)$, depend on the level of soil quality, S and an action, x. The soil quality changes in a deterministic manner according to $S_{t+1} = f(S_t, x_t)$. The value of S at the start of the program is given.

a) Write down the optimal control problem (the statement of the objective and the constraints)

b) Write down the dynamic programming equation (DPE).

c) Use the DPE to obtain the Euler Equation.

d) Write down the equations that determine the optimal steady state (assuming it exists and is interior).

4) More details on the resource extraction problem. Utility in a period is the log of harvest, h: $U = \ln(h)$. The stock evolves according to $S_{t+1} = (S_t - h_t)^{\alpha}$. The discount factor is β. The time horizon is $T < \infty$.

a) Write the optimal control problem

b) Write the DPE and the boundary condition for the DPE (i.e. the value function at T).

c) Use a inductive proof to show that for all $t \leq T$ the value function is of the form $A_{0\tau} + A_{\tau} \ln(S_{\tau})$, where $\tau \equiv T - t$, the "time to go" (until the end of the problem.)

d) Write down the difference equations that $A_{0\tau}$ and A_{τ} satisfy. How does the control rule change over time? (Do you extract a larger or a smaller fraction of the stock?) Obtain sufficient conditions on the parameter values to insure that the steady state of the equations is stable.

e) Let $\tau \to \infty$ to obtain the steady state control rule. (Find the steady state of the difference equations you derived in part d.) How is this related to the control rule we obtained when we started with an infinite horizon problem?

5) A lobby group is able to affect the probability of election of a friendly party. Elections are held every period. When the friendly party is in office, the group receives a flow of rents b. It receives 0 rents when the unfriendly party is in office. The lobby group contrinbutes c_t in period t. The probability of reelection of the friendly party is $p(c)$ (given that it is currently in power) and the probability of reelection of the unfriendly party is $q(c)$ (given that it is currently in power). The group's discount factor is β. The group wants to

maximize the expectation of the present discounted value of the stream of future utility. Write down its control problem and the corresponding DPE.

6) (Bonus points) We have encountered two problems (two combinations of single period payoffs and equation of motion) for which the optimal control is a linear function of the state. It probably seems that there must be other problems (combinations of single period payoffs and equation of motion) for which the optimal control rule is linear. How would we go about identifying these problems?

If we knew how to do this, we could generalize the method to find problems that would yield other particular control rules. In other words, instead of beginning with the primitive functions (the single period payoff and the equation of motion) we might begin with a particular control rule (not necessarily linear) and find the primitive functions that imply that this particular form of control rule is optimal. This procedure can be useful for generating classes of control problems for which we can obtain closed form solutions. It might also be useful for empirical work. For example, the method gives us restrictions on the functional form of primitive functions, associated with particular functional forms of the control rule.

This problem takes you through the steps of deriving these restrictions. We will work with a continuous time autonomous control problem. The state is x and the control is y; the flow of utility is $U(x, y)$ and the equation of motion is $\dot{x} = F(x, y)$. The instantaneous discount rate is r.

a) Write down the optimal control problem.

b) Write down the dynamic programming problem. (This is an autonomous problem, so you want to find a value function that depends only on the state - rather than on the state and time.

c) Use the method described in the notes to obtain a differential equation that the optimal control rule must satisfy, that is, the equation for dy/dx . (Of course, you could also use the Maximum Principle to find this differential equation - but this is a problem set on DP.)

d) Now assume a particular control rule. To keep life simple, take a linear control rule $y = ax$, where a is a parameter to be determined. Substitute this control rule into the differential equation you previously determined. The result is an equation involving U, F and their derivatives. The primitive functions must satisfy this equation in order for the linear control rule to be optimal.

e) Part d gives us a necessary condition on the primitive functions - we derived that condition using the first order condition for maximization of the right hand side of the DPE. What is the second order condition for this maximization? What restriction must the primitive function satisfy in order for the linear control rule to be optimal?

f) From now on we'll work only with the necessary condition from part d. We still have a mess - nothing that looks useable. Confronted with such a situation, what do economists do? We make assumptions! There are all kinds of assumptions we might make. Here are some of the most obvious.

i) Assume that U depends only on the control y and that F is separable: $F_{xy} = 0 \Longrightarrow F = f_1(x) + f_2(y)$ for some functions f_i. Plug these assumptions into the expression you obtained in part d.

ii) Still a mess? Make more assumptions. Suppose that $U(y) = y^\alpha$ and that $f_2(y) = y$. Plug these additional assumptions into the expression you obtained from above, and solve for the unknown function $f_1(x)$.

g) Review. What have we accomplished? We know that if $U(y) = y^\alpha$ and $\dot{x} = f_1(x) + y$, where $f_1(x)$ is a particular function whose form we have determined, then the linear control rule $y = ax$ (for some constant a) satisfies the necessary conditions to solve the control problem. (Now we should go back and check the second order condition.) If we had made other assumptions, we would have obtained other primitive functions.