Notes on Krugman's model. (File title "Migration")

This note has the following three objectives:

1) Make sure you understand why the upper limit of integration in the definition of q is T rather than infinity.

2) Explain what it means to "reverse time" in order to solve the system of ODEs, and why I want to do that. At time 0 I know the value of L (this is given by history) but I do not know the value of q. At time T (whatever that calendar date happens to be) I know that q=0 and that L=0 or L=1. Thus, if I solve the problem backwards in time, using two sets of "initial condition" q=0, L=0 and q=0, L=1, I can figure out the value of L at any "time-to-go". This procedure enables me to find out how long it would take the economy to move from a given initial condition to a particular boundary value (e.g., all the labor in manufacturing), and in the case of indeterminacy, it gives me a simple way of finding out how large the region of indeterminacy is.

3) Show you how you can find this information numerically, using Scientific Workplace. (No home should be without one.

1) Suppose I know my demand for BART tickets and I also know the future price of BART tickets. I learn that in the year 2005 riding BART will be free (e.g., this is part of an effort to get cars off the road). I can buy a BART pass that will let me use BART for free right now, and this pass is good at any time, and in perpetuity. How much should I be willing to pay for this pass? It should be obvious that I would calculate the present discounted value of riding BART for free from now until the year 2005. This does not mean that after the year 2005 I derive no benefits from riding BART. It simply means that I should not include those benefits in my calculation of the value of the pass, because I will not have to pay for those benefits in any case. In Krugman's model, an instant after T (when, by definition, migration is 0) any individual can migrate at 0 cost.

2 The original system is

$$\begin{aligned}\frac{dq}{dt} &= rq-(\alpha+bL)\\ \frac{dL}{dt} &= \frac{q}{\gamma}\end{aligned}$$

where $\alpha=a-c<0$. The unstable interior steady state solves $\alpha+bL=0$; define the steady state as $s=\frac{-\alpha}{b}$.

Define $\tau = T - t$, so $d\tau = -dt$. τ is the amount of time it takes to reach a boundary ($L = 0$ or $L = 1$), given a value of L at time t. If I reverse time, I write the system as

$$\begin{aligned} \frac{dq}{d\tau} &= -rq + (\alpha + bL) \\ \frac{dL}{d\tau} &= -\frac{q}{\gamma} \end{aligned}$$

.My new "initial condition" for this system is the boundary $q = 0, L = 1$ (Of course I could also have looked at the boundary $L = 0$.)

3) If I want to solve this system numerically, I need parameter values. I'll define the following values

$\alpha = -.5$

$r = .1$

$b = 1$

$\gamma = 10$

So the steady state is

$s = .5$.

For these parameter values the discriminant is

$\Delta = r^2 - \frac{4b}{\gamma} = -.39$

So I know that the interior (unstable) equilibrium is a spiral point. To solve the reversed system with these parameter values –and the assumption that all labor end in the manufacturing sector, I need to solve

$$\begin{aligned} \frac{dq}{d\tau} &= -rq + (\alpha + bL) \\ q(0) &= 0 \\ \frac{dL}{d\tau} &= -\frac{q}{\gamma} \\ L(0) &= 1 \end{aligned}$$

, Functions defined: q, L

With a single keystroke, ScientificWorkplace gives me the numerical solution (not shown) which I ask it to graph in Figure 1. The horizontal axis is τ.

(The thick solid line is the graph of q and the thin dashed line is the graph of L.

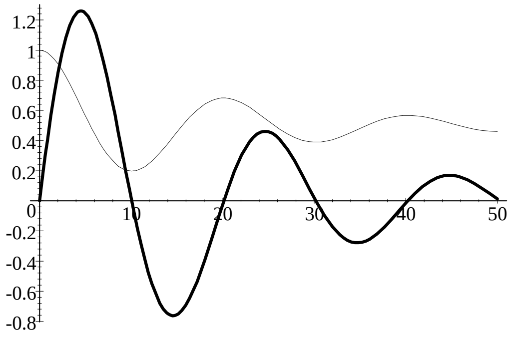

Figure 1:

Note that when q changes signs, L changes directions. The graph shows that the smallest initial value of L which is consistent with L eventually reaching the boundary $L = 1$ is approximately $L = .2$. If L begins at that value, it takes about 10 units of time to reach the boundary.

L, q

Figure 2 shows the phase portrait of L,q space. (The figure shows only the trajectory that converges to L=1 – not the isoclines.) Notice that the domain of the spiral is from about 0.2 to 1, and that the origin of the spiral is the unstable steady state, .5. If the initial value of L is anywhere in this domain, there is a competitive rational expectations equilibrium that takes all labor to the manufacturing sector. This trajectory could be monotonic, or it could be non-monotonic, depending on which arm of the spiral the economy begins on.

Now I'll change parameter values to show the case where the unstable interior equilibrium is a node (rather than a spiral). I will increase the convexity of adjustment cots by increasing γ and decrease the importance of the future by increasing r

$\gamma = 100$

$r = .8$

$\Delta = r^2 - \frac{4b}{\gamma} = .6$

(Note that a discount rate of $r = .8$ does not necessarily imply that the discount rate is "unreasonably large"makes sense. The magnitude of r

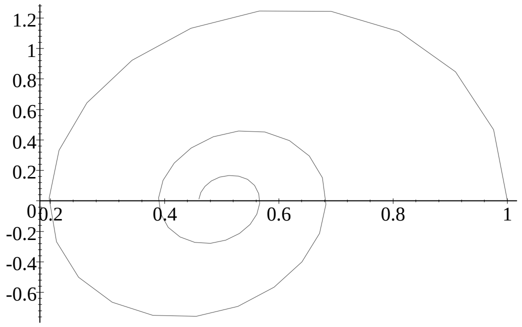

Figure 2:

depends on the units of time. Make sure that you understand this point:. The discount factor for one unit of time is e^{-r}. Suppose for example that I define one unit of time to be 10 years. Then the discount factor for one year is $e^{-\frac{r}{10}} = .923\,12$, or a yearly discount rate of less than 8%. Without knowing the definition of a unit of time, the value of r tells you nothing.)

$$\begin{aligned}
\frac{dq}{d\tau} &= -rq + (\alpha + bL) \\
q(0) &= 0 \\
\frac{dL}{d\tau} &= -\frac{q}{\gamma} \\
L(0) &= 1
\end{aligned}$$

, Functions defined: q, L

Figure 3 shows the graph of L and q as functions of time. Again, the graph of q is the solid thick line and the graph of L is the thin dashed line. This graph shows that it takes about 100 units of time for L to travel from approximately .7 to 1. q is positive along the entire trajectory, so L is monotonic. (As time goes forward, L increases. This graph shows q and L as functions of τ, reversed time, so L is decreasing.

The phase portrait (figure 4) is

q, LIn drawing this phase portrait I set the number of periods to 300. Figure 5 shows another phase portrait with the number of periods equal to

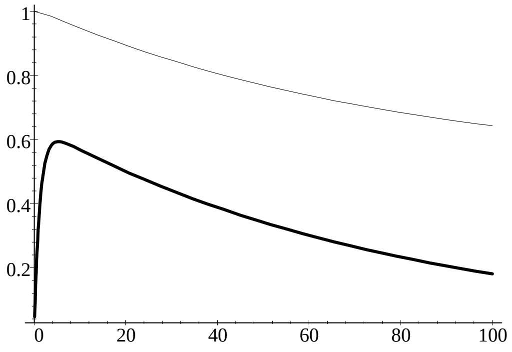

Figure 3:

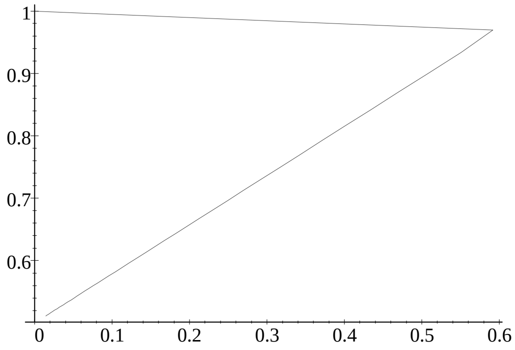

Figure 4:

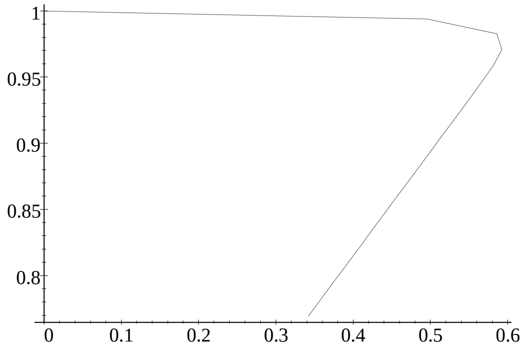

Figure 5:

50.

q, LIf the initial value of L is close to the interior steady state, e.g. about .55 as in the first phase portrait, it takes 300 periods to reach L=1. If the initial value of L is far from the interior steady state, e.g. about .75 as in the second phase portrait, it takes about 50 periods to reach L=1. Near the steady state, the system is moving very slowly. It takes infinitely many periods to "reach" the steady state.