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Notes for Dynamics

IX. Linear Control Problems

1) One-state variable linear control problem.
2) Necessary and sufficient condition for optimality.
3) Most Rapid Approach Path (MRAP), and singular arc.
4) Fishing example.
5) Two-state variable problem, a durable non-renewable resource with decay.

K & S Section I.16 and II.13, Clark Ch 2.7

General 1 state variable problem

(1) $$\max \int_{t_0}^{t_1} \left[G(t,x) + H(t,x)\dot{x} \right] dt$$

$$\text{s.t.} \quad A(t,x) \le \dot{x} \le B(t,x), \quad x_{t_0}, x_{t_1} \quad \text{given}$$

$$\text{Euler equation} \quad \left(F_x = \frac{d(F_{\dot{x}})}{dt} \right)$$

$$G_x + H_x\dot{x} = \frac{d}{dt}H = H_t + H_x\dot{x}$$

(2) $$G_x = H_t$$

Any x^* that satisfies (2) is singular solution (arc)

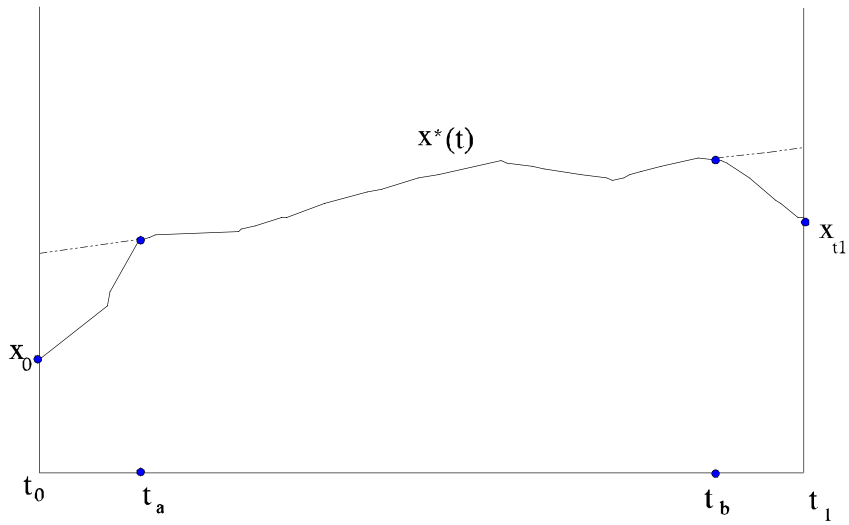
Thrm. Suppose $\dfrac{\partial G}{\partial x} \gtrless \dfrac{\partial H}{\partial t}$ whenever $x \gtrless x^*(t)$.

The optimal solution is

(i) If $x_0 > x^*$ (t_0), set $\dot{x}(t) = A(\;)$ until x^* hit
If $x_0 < x^*$ (t_0), set $\dot{x}(t) = B(\;)$ until x^* hit

Define t_a as time you hit x^* using (i)

(ii) Analogous description at RHS, with t_b as time you leave singular path.



over t_0, t_a, set $\dot{x} = B$, over t_b, t_1, set $\dot{x} = A$. (MRAP) over t_a, t_b, solution follows singular path.

Example from fisheries (Clark). This problem is autonomous.

(3) $$\max \int_0^{\infty} e^{-rt}\left(p - c(x)\right)h \, dt$$

s.t.

(4) $$\dot{x} = F(x) - h$$ x_0 given, $0 \le h \le h_{\max}$

Exercise. Eliminate h to write (3) and (4) so it looks like (1). Check to make sure that assumptions of Thrm are satisfied.

Solve (3) and (4) using maximum principle.

$$H = \left(p - c(x)\right) h + \lambda\left(F(x) - h\right)$$

F.O.C.

(5)
$$\frac{\partial H}{\partial h} = p = c(x) - \lambda \begin{cases} < 0 \Rightarrow h = 0 \\ = 0 \Rightarrow h = h^* \\ > 0 \Rightarrow h = h_{\max} \end{cases}$$

(6)
$$\dot{\lambda} = r\lambda + c'(x)h - \lambda F'(x)$$

Suppose we have an interior sol'n: $h = h^*$, then differentiate mid equation (5), use (6) and (4) and (5)

$$0 = -c'(x)\dot{x} - \dot{\lambda} = -c'(x)\left(F(x) - h\right) - \lambda\left(r - F'(x)\right) - c'(x)h$$
$$= -c'(x)\left(F(x) - \underline{h}\right) - \left(p - c(x)\right)\left(r - F'(x)\right) - \underline{c'(x)h}$$

(7)
$$\left[p - c(x)\right]F'(x) - c'(x)F(x) = r\left(p - c(x)\right)$$

I could have gotten (7) by recognizing that "singular arc" is a point, so has to be a steady state. Set (4) and (6) = 0, use (5) to obtain (7).

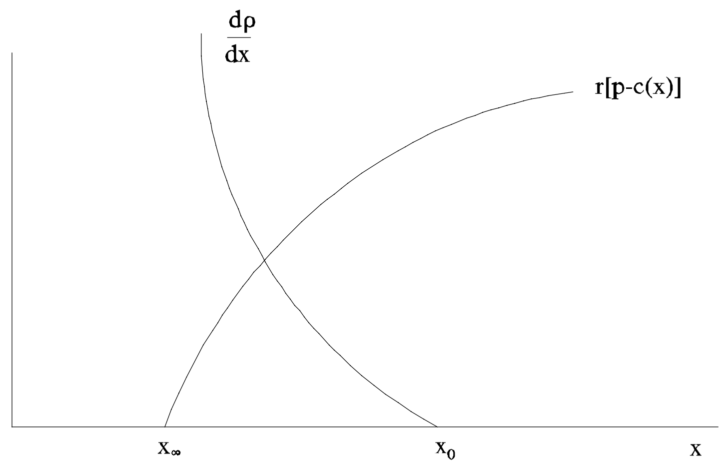
Interpret (7). Define $\rho = \left[p - c(x)\right]F(x) \equiv$ "sustained economic rent"

Rewrite (7) as

(8)
$$\frac{1}{r}\left[\frac{d\rho(x)}{dx}\right] = p - c(x)$$

An extra unit of harvest yields current revenue of p - $c(x)$, but flow of future loss is $\frac{d\rho}{dx}$.

Discount this to obtain LHS of (8)



$$\frac{d^2\rho}{dx^2} = -2c'F' + (p - c(x))F'' - c''F$$

(Even with concave *F*, can't guarantee unique soln of (8). I've assumed it.)

x_0 corresponds to rent maximizing stock (r = 0)
x_∞ corresponds to competitive equilibrium ($r = \infty$)

A two state variable problem durable, nonrenewable resource (e.g. iron, not oil). (Notes follow Karp, Monopoly Power can be Disadvantageous in the Extraction of a Durable Nonrenewable Resource.) The point of this exercise is to show you how to find and interpret the singular arc in a two state variable autonomous control problem. Here the arc is 1-dimensional. In the one-state variable problem, the arc is 0-dimensional, i.e. a point.

(1) $\dot{S} = -m$, S_0 given (resources stock)

(2) $\dot{Q} = m - \delta Q$, Q_0 given (durable good stock)

$U(Q)$ = utility of stock of good
$c(S)m$ = extraction cost.

Social planner's problem

(3) $$\max \int_0^{\infty} e^{-rt}\left[U(Q) - c(S)m\right] dt$$

s.t. (1) and (2)

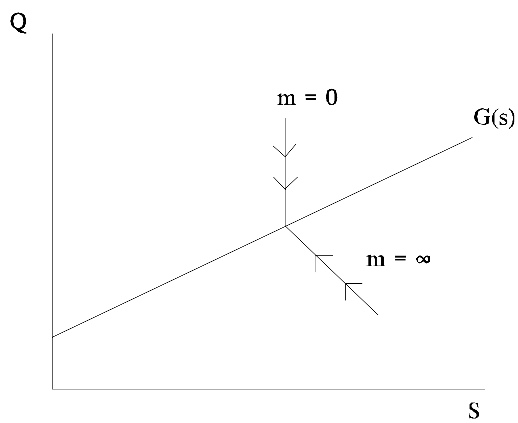
$$H = U(Q) - c(S)m - \lambda m + \eta(m - \delta Q) = U - \eta\delta Q + (\eta - \lambda - c(S))m$$

λ = S.V. of S, η = S.V. of Q

(4) $$\frac{\partial H}{\partial m} = -c(s) - \lambda + \eta = 0 \quad \text{(on singular arc)}$$

($\eta - c - \lambda$ is called the "switching function")

(5) $$\dot{\lambda} = r\lambda + c'(s)m$$



(6) $$\dot{\eta} = (r + \delta)\eta - U'(Q)$$

[Exercise: integrate equation 6 (use integrating factor) to obtain the expression $\eta_t = \int e^{-(r+\delta)\tau} U'(Q_\tau) d\tau$, where limits of integration are t, ∞. Interpret this expression.]

Find singular arc.

Define $U'(Q) \equiv F(Q)$ = rental of a unit of the durable good. η is the price that buyers of the good would be willing to pay. (6) says capital gains + dividends = opportunity cost of purchase + depreciation.

Differentiate (4)

$$\begin{aligned} \dot{\eta} &= c'(S)\dot{S} + \dot{\lambda} && \text{use (5)} \\ &= -c(S)m + r\lambda + c'(S)\,m && \text{use (4)} \end{aligned}$$

(7) $$= r\left(\eta = c(S)\right) \qquad \text{(The Hotelling rule!)}$$

Equating (7) and (6)

$$(r + \delta)\eta - F = r\left(\eta - c(S)\right)$$

(8) $$\eta = \frac{\left[F - rc(S)\right]}{\delta}$$

(8) gives equil. goods price on singular arc.

Define singular arc as $Q = G(S)$. Differentiate *wrt* time using (1) and (2)

(9) $$m = \frac{\delta G}{\left(1 + G'\right)}$$

Differentiate (8), use (6), (8) and (9) to get

(10) $$rF\left(G(S)\right) - r\left(r + \delta\right)c(S) = -F'\left(G(S)\right)\delta G(s) + \frac{\left[rc'(S) + F'\left(G(S)\right)\right]\delta G(S)}{\left(1 + G'(S)\right)}$$

This is an ODE in S, solve to find singular arc (need B.C.)

e.g. if

$F = a - bQ$, $C(S) = k_0 - kS$, with $\frac{a}{r + \delta} < k_0$ then $G(s) = gS$, where g is positive solution to a quadratic.