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Notes for Dynamics

VIII. "Nonconvex" Control Problems

1) Describe growth model that leads to non-convex control problem
2) Sketch phase portrait of solutions to FOCs
3) Identify optimal candidate.
4) Economic interpretation.
5) A pollution control problem with similar features.
6) Sketch phase portrait
6) A different way to identify optimal trajectory.

In this section we study a control problem with increasing returns to scale (IRTS) of the state equation over some interval. For some initial conditions it is optimal to drive the state to 0 (e.g., wipe out the resource) and for some initial conditions it is optimal to build up the state. In this problem typicaly ∃ multiple solutions to FOC's, and SOC's do not hold. How do you identify the optimal solution?

<u>1) Growth model with IRTS</u> (Brock and Malliaris, pp 159 - 168)

2 sector optimal growth model

x = capital stock, $\dot{x}$ = investment
x_1 = amount of capital in "neoclassical" sector (concave production function)
y_1 = production of power (an intermediate good)
$g_1(x_1, y_1)$ = output of capital / consumption good, a neoclassical prod'n function
$y_1 = g_2(x\text{-}x_1)$ increasing and convex. Economy can allocate capital between sectors

$U(g_1(x_1, y_1) - \dot{x})$ = utility

objective: $$\max \int_0^\infty e^{-\rho t} U\big(g_1(x_1, y_1) - \dot{x}\big) dt$$

Define

(1) $$g(x) \equiv \max_{x_1} g_1\big(x_1, g_2(x - x_1)\big)$$

This is maximum output of final good, given x. Assume $g(x)$ convex-concave (M & B show this is the case).

Rewrite problem as

$$\max \int_0^\infty e^{-\rho t}\, U(c)\, dt$$

$$c + \dot{x} = g(x) \qquad x_0 \text{ given}$$

We will show that if initial level of capital is low, it is optimal to run it down (stagnate). If initial level is sufficiently high, it is optimal to build it up (save).

We begin by looking at the characteristics of the function g(x)

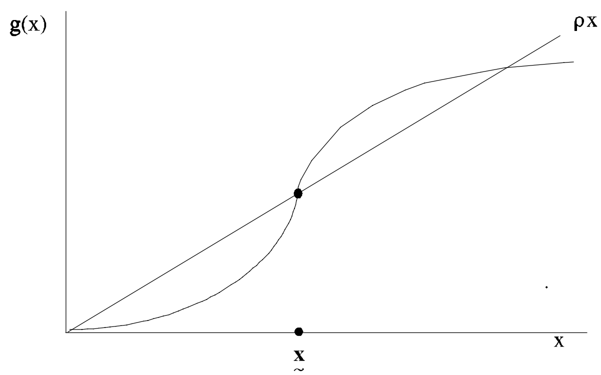

Figure 1

$\underset{\sim}{x}$ defined as lowest value of x where $\rho(x) = g(x)$

FOC to (1):

$$x_1^0 \text{ solves (1)} \Rightarrow g_{1x1} - g_{1y1}g_2' = 0$$

$$\text{envelope thrm} \Rightarrow g' = g_{1y1}g_2' = g_{1x1}$$

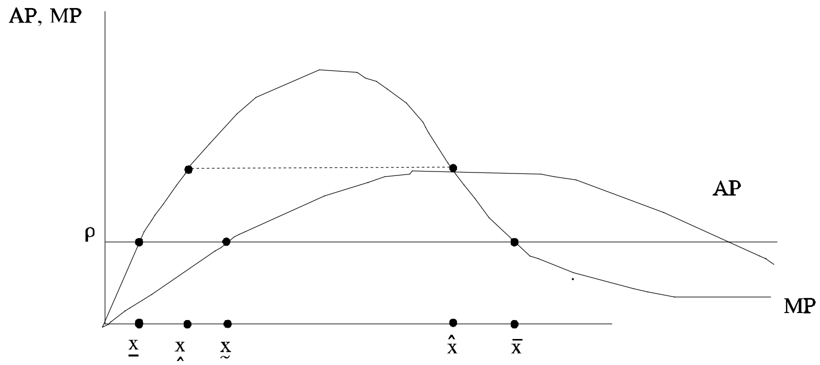

Figure 2

$$AP = \frac{g(x)}{x} \qquad MP = g'(x)$$

Now we begin to look at the control problem. Here are the Hamiltonian and necessary conditions.

(2) $$H = U(c) + \lambda\left(g(x) - c\right) \qquad \text{(current value)}$$

(3) $$U'(c) = \lambda \quad \Rightarrow \quad c = C(\lambda) \qquad C' = \frac{1}{U''} < 0$$

(4) $$\dot{\lambda} = \lambda\left(\rho - g'(x)\right)$$

(5) $$\dot{x} = g(x) - C(\lambda)$$

2. Sketch phase portrait We will sketch the portrait in x, λ space.

First we find the isocline(s) for $\dot{\lambda} = 0$.
$\dot{\lambda} = 0 \Rightarrow \rho = g'(x) \Rightarrow x = \underline{x}$ or $x = \bar{x}$.
for $\underline{x} < x < \bar{x}$, $g'(x) > \rho \Rightarrow \dot{\lambda} < 0$. These results are summarized in figure 3.

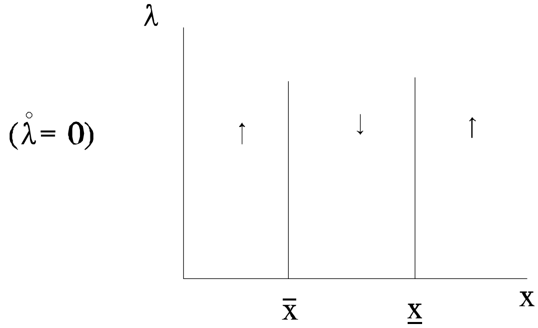

Figure 3

Now we find the isocline for $\dot{x} = 0$.

$$\dot{x} = 0 \Rightarrow \quad g(x) - C(\lambda) = 0 \Rightarrow \; g'(x)dx - C'd\lambda = 0$$

$$\frac{d\lambda}{dx}\Big|_{\dot{x}=0} = \frac{g'(x)}{C'} \quad (< 0 \text{ when } g'(x) > 0 \text{ since } C' < 0\text{; see eqn 3})$$

So we know that the isocline for $\dot{x} = 0$ is downward sloping, as shown in figure 4. Now we have find the directional arrows; i.e., is x increasing or decreasing above the isocline? To determine this, we hold λ fixed, increase x when on isocline. In other words, we evaluate the derivative $[d\dot{x}/dx]_{\dot{x}=0} = g'(x) > 0 \Rightarrow$ sign $(\dot{x})$ = sign $g'(x) > 0$. In other words, x is increasing above the isocline, as the arrows show in figure 4.

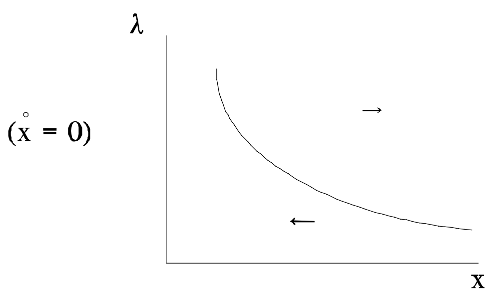

Figure 4

Suppose that $U'(0) = \infty$. This assumption implies that $\dot{x} = 0$ isocline is asymptotic to the λ axis: When $x = 0$, $g(0) = 0$, so if both $\dot{x} = 0 = x$, it must be the case that $C = 0$. From equation (3), it must then be the case that $\lambda = \infty$.

Now superimpose figure 4 on figure 3, to obtain figure 5.

We see that there are two steady states, at $\underline{x}$ and $\bar{x}$. Label the trajectory that converges to the SS at $\bar{x}$ I. Label the trajectory that spirals out of the steady state at $\underline{x}$ and approaches x = 0, λ = ∞, as II. (Figure 5)

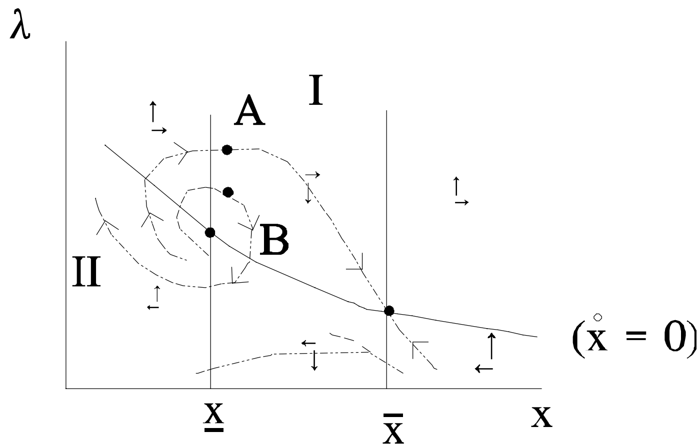

Figure 5

Now consider characteristics of the two steady states.

$\underline{x}$ is unstable (pick any point in neighborhood of $\underline{x}$ and see that you don't approach SS.)

$\bar{x}$ is a saddle point. Intuition: SS must be on some tragectory. That trajectory converges to SS.
(You can use the geometric device on page 3:14, or the linearization method, to prove that $\bar{x}$ is a saddle point.)

3. Identify optimal trajectory

First, we identify *candidates* for optimal trajectory, i.e. those which solve all necessary conditions.

Trajectories I and II are the only candidate trajectories that satisfy all the necessary conditions, including the transverality conditions. Consider how we rule out other trajectories in phase space. (Those trajectories satisfy the necessary conditions (3) - (5), but they do not satisfy transverality condition.) For example, any trajectory "between" I and II (such a trajectory eventually lies above trajctory I) approaches x = ∞ and λ = ∞, so $\lim_{t\to\infty} e^{-rt}\lambda(t)x(t) \neq 0$. The intuition for why such a path cannot be optimal is obvious: consumption approaches 0 (equation 3). This means that the stock of capital grows without bound, and consumption goes to 0, which cannot be optimal. A trajectory "below" II hits the λ axis in finite time, T. The transversality condition in this case is H(T) = 0. This condition and (2), together with x = 0, implies that U(c)/c = λ. However, that equation is not consistent with (3) in view of the concavity of U.

So we have ruled out trajectories other than I and II using transversality condition. We need to show that I and II satisfy that condition.

Trajectories I and II in Fig 5 satisfy lim $t \to \infty$ $e^{-\rho t}$ $\lambda(t) = 0$ (TC)
(Obvious for I because λ_∞ is finite. For II use (4) ⇒ λ growing at rate < ρ.)

Now we have the only two candidates, and we have to find which trajectory is optimal. The reason why this is not straightforward is that there is an interval of initial conditions, (x_*, x^*) in figure 6, for which both candidates are feasible. Outside that interval, only one candidate is feasible.

The point x_* is the smallest value of x on trajectory I. x^* is the largest value of x on trajectory II.

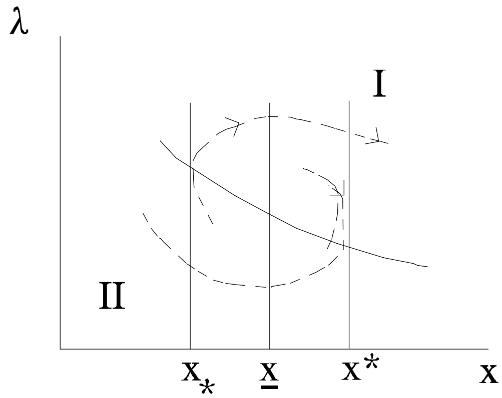

Figure 6

We will use a lemma to find out how to choose the correct trajectory for an initial condition in the interval (x_*, x^*).

Define $V_{II}(x_0)$ and $V_I(x_0)$ as payoff when trajectory II or I is followed.

Recall DPE

(6) $$\rho V = \max_c H \quad \text{(Hamiltonian)}$$

By (2) and (3)

(7) $$H = U(c) + U'(c)\ \left(g(x)\ -\ c\right) \quad \text{for optimal } C.$$

Lemma: $V_{II} > V_1$ iff

(i) $$\dot{x}_1(0) \geq 0 \text{ and } U'\left(C_1(0)\right) < U'\left(C_2(0)\right)$$

or

(ii) $$\dot{x}_1(0) \leq 0 \text{ and } U'\left(C_1(0)\right) > U'\left(C_2(0)\right)$$

Proof

(8) $$\rho(V_{II} - V_I) = U(c_2) - U(c_1) + U'(c_2)\big(g(x_0) - c_2\big) - U'(c_1)\big(g(x_0) - c_1\big)$$

Iff (i) or (ii) hold then the expression on the RHS of (8) is greater than or equal to:

(9) $$U(c_2) - U(c_1) + U'(c_2)\big(g(x_0) - c_2\big) - U'(c_2)\big(g(x_0) - c_1\big)$$

(e.g. if (i) holds I've subtracted a larger number in last term.) Now, by concavity of U, the expression in (9) is greater than or equal to

$$\geq U'(c_2)(c_2 - c_1) + U'(c_2)\big(g(x_0) - c_2\big) - U'(c_2)\big(g(x_0) - c_1\big) = 0 \quad \bullet$$

Apply lemma to show $V_{\mathrm{I}}(x^*) > V_{\mathrm{II}}(x^*)$ and $V_{\mathrm{II}}(x_*) > V_{\mathrm{I}}(x_*)$.

e.g. at x^*, $\dot{x}_1 \geq 0$, and $\lambda_1 > \lambda_2$, so by (3), $U'\big(c_2(0)\big) < U'\big(c_1(0)\big)$. Therefore condition (i) is not satisfied), so $V_{\mathrm{I}}(x^*) > V_{\mathrm{II}}(x^*)$.

at x_*, $\dot{x}_1(0) \leq 0$ $U'\big(c_1(0)\big) > U'\big(c_2(0)\big)$ so condition (ii) is satisfied.

We want to compare the graphs of the value functions. We have comparison at two points, x_* and x^*. To complete the comparison need to show that the value functions are monotonic in x, as shown in figure 7.

Slope of V_i given by $\lambda_i = U'(c_i)$. From phase diagram note that $\lambda_1 > \lambda_2$ over (x_*, x^*), so graph is as shown in fig 7.

Conclude for $x_0 \leq x_s$, optimal to run down capital (stagnate).

Get convex increasing case by pushing large steady state to ∞. Get concave (usual) case by pushing small SS to 0.

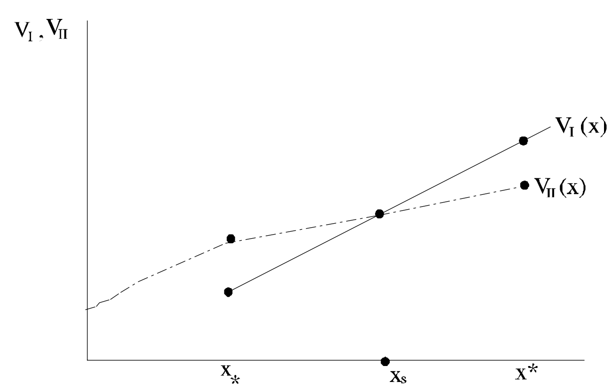

Figure 7

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(Now we study an environmental problem that has a mathematical structure similar to the above. Here notes follow Tahvonen and Salo.)

$$\max \int_0^\infty e^{-\delta t}\left[U(e) - D(z)\right]dt$$

(1) $$\dot{z} = e - a(z)$$
$$0 \le e \le \hat{e}$$

$U'(0)$ finite, U is concave, D is convex.

$a(z)$ concave - convex $\qquad U'(\hat{e}) = 0$

$a(z) = 0$ for $z \ge \bar{z}$, pollution is irreversible when stock exceeds $\bar{z}$.

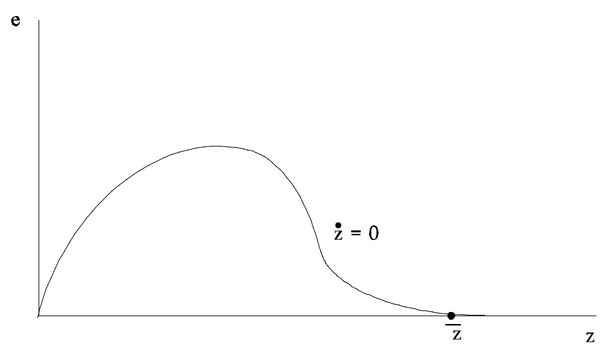

Figure 8

λ = C.V. costate, H = C.V. Hamiltonian. We will use FOC's to find the $\dot{e} = 0$ isocline, and superimpose the result on figure 8 - see figure 9.

$$H = U(e) - D(z) + \lambda\left[e - a(z)\right]$$

F.O.C.

(2) $$\dot{\lambda} = D'(z) + \left(\delta + a'(z)\right)$$

$$(3) \qquad e = \begin{cases} \hat{e} \text{ if } \lambda \geq 0 \\ e^* \text{ if } -U'(e^*) = \lambda & \text{(interior soln.)} \\ 0 \text{ if } -U'(0) > \lambda \end{cases}$$

For interior soln, differentiate (3), use (1) and (2) $\Rightarrow$

$$(4) \qquad \dot{e} = -\left[D'(z) - U'(e)(\delta + a'(z)) \right] / U''$$

$$(5) \qquad \dot{e} = 0 \Rightarrow D'(z) - U'(e)(\delta + a'(z)) = 0$$

$$(6) \qquad \frac{de}{dz}\Big|_{\dot{e} = 0} = \left[D''(z) - U'(e)a''(z) \right] / U''(e)\left[\delta + a'(z) \right]$$

Various possibilities:

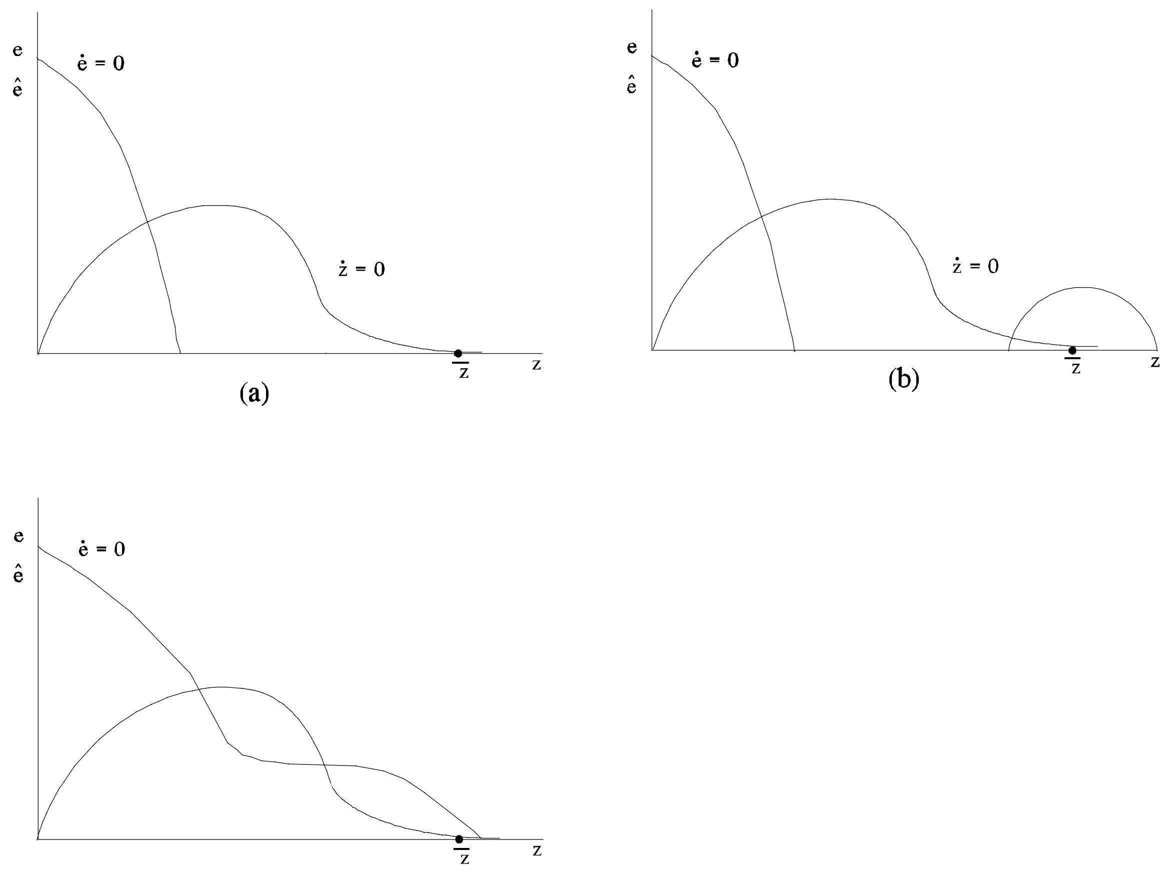

Figure 9

Increase in δ shifts $\dot{e} = 0$ isocline up, so graph could change from *a* to *b* to *c*. $\dot{e} = 0$ isocline defined for $a' > -\delta$. There isocline has negative slope, and above iscoline, $\dot{e} > 0$.

Characteristic of steady states.

They show that SS's with lowest and highest z are saddle points. Intermediate points are unstable nodes or unstable focus.

Consider portrait like *9c*, where there are three steady states.

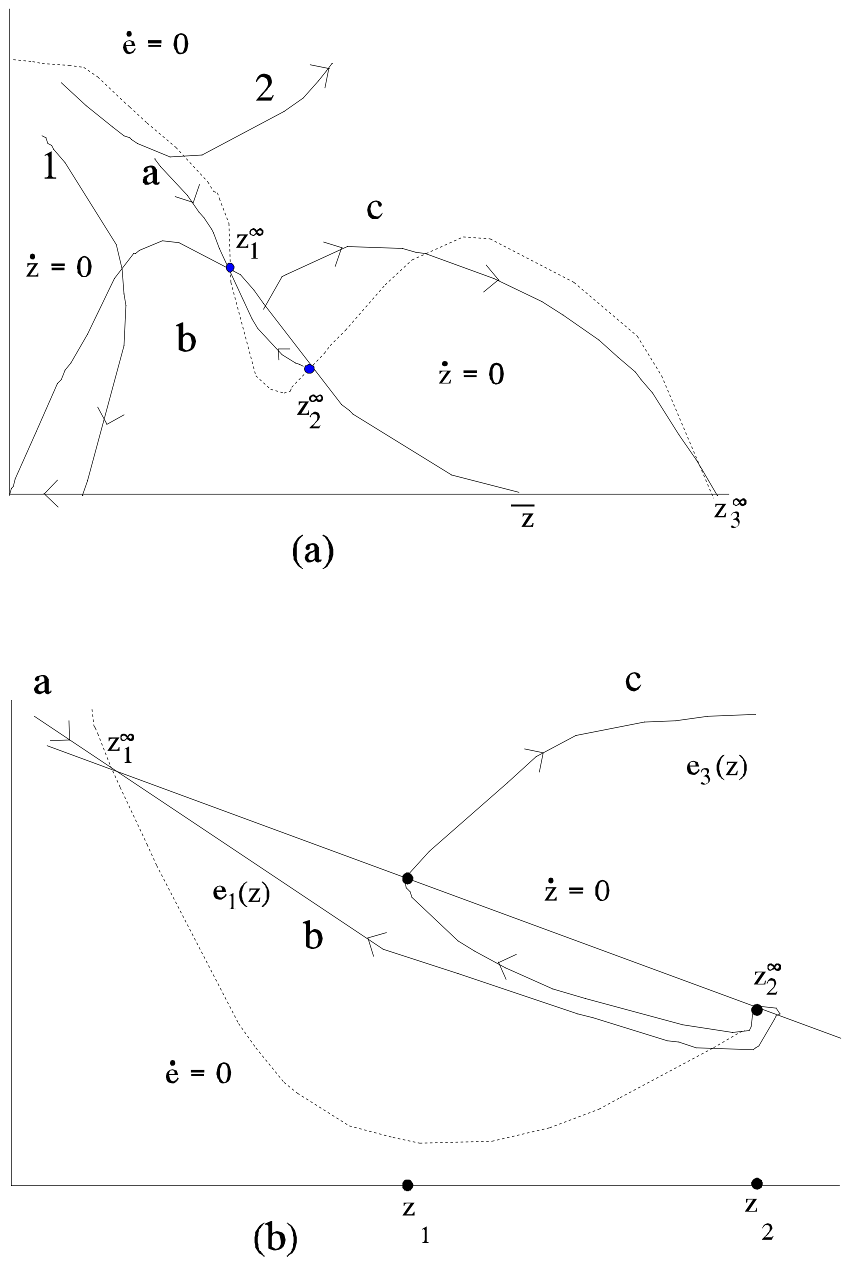

Figure 10

z_1^∞ and z_3^∞ are saddle points.
z_2^∞ is unstable focus.

(a) and (b) are saddle paths.

Rule out path like (1) and (2)

Along (1) $e \to 0$ and $z \to 0$ so $U' \to +$ but $D' \to 0$, so can increase utility by consuming more in steady state.

Along (2) $e \to \hat{e} \Rightarrow U' \to$ zero, but $z \to \infty \Rightarrow D' \to \infty$, so can increase utility by consuming less in steady state.

We're left with paths that converge to z_1^∞ or z_3^∞, or stay at z_2^∞ (if we start there). Use DPE at time 0, given state z_0.

The lower pannel of figure 10 is an enlargement of the mid region of the top pannel. The curve $e_1(z)$ is the trajectory that converges to z_1^∞ and the curve $e_3(z)$ converges to z_3^∞.

Recall the relation between the dynamic programming equation and the Hamiltionian: $\delta J = \max_e H(e,z,\lambda)$. Assuming an interior solution, we can use the FOC (3) to eliminate λ and write

(7) $$\delta J^i(z_0) = U(e_o^i) - D(z_0) - U'\left[e_0^i - a(z_0)\right] \equiv F(e_0^i, z_0)$$

e_i^0 tells us whether we begin on the path that takes us to z_1^∞ (i = 1) or z_3^∞ (i= 3). Both of these satisfy the FOC (3). I've denoted the function on the RHS of (7) as F, rather than H, the Hamiltonian, to remind you that it is the maximized Hamiltonian.

(8) $$\frac{\partial F}{\partial e_0^i} = -U''(e_0^i)\left(e_0^i - a(z_0)\right) \begin{cases} < 0 \text{ for } 0 < e < a(z) \\ > 0 \text{ for } \hat{e} > e > a(z) \end{cases}$$

These inequalities tell us that, for given z, we increase the payoff by moving to another path (which also solves the FOCs) which is further from the $\dot{z} = 0$ curve. E.g., if we begin where $\dot{z} < 0$, we increase payoff by decreasing e.

Remember that the costate variable, which by (3) equals -U'(e), is the shadow value of the stock.

(9) $$\frac{\partial F^i}{\partial z_0} = -U'\left(e_0^i\right)$$

Define z_1 lowest stock on path c, and z_2 highest stock on path b.

Compare value of programs at $z_0 = z_1$.

At z_1, $e_0^1 < a(z_1)$ and $e_0^3 = a(z_1)$, so $e_0^1 < e_0^3$. By (8) we have

(10) $$F(e_0^1, z) > F(e_0^3, z_1)$$

at $z_0 = z_2$, $e_0^1 = a(z_2)$ and $e_0^3 > a(z_2)$ so by (8) we have

(11) $$F(e_0^3 z_2) > F(e_0^1 z_2)$$

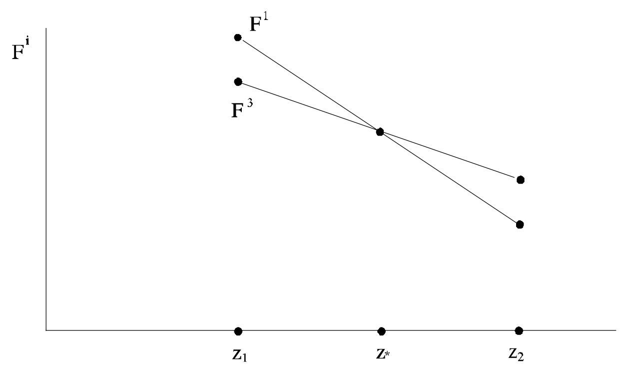

Figure 11

I've used (10) and (11) to draw endpoints of graphs and (9) to conclude they are downward sloping. Also by (9) and concavity of U, curve is flatter the greater is e. Since e is always greater on path c than path b, conclude picture is as shown, ∃ unique intersection, z^*

$$\Rightarrow \quad z_0 > z^* \text{ go to } z_3^\infty$$
$$z_0 < z^* \text{ go to } z_1^\infty.$$