

VII. Limit Cycles in Intertemporal Adjustment Models

1. Describe basic models of convex adjustment, with one state variable.
2. Review basic fish problem with no costs of adjusting control.
3. Add adjustment costs to basic resource (e.g. fishing) problem, to get two-state variable problem.

1) What is an "adjustment model"?

(Recall example of Krugman's paper in first set of notes.) Standard example, costly adjustment of capital.

$\pi(K)$ = restricted profit function $C(I)$ = investment cost

$$C', C'' > 0 \quad \text{e.g.} \quad C = wI + \frac{\gamma I^2}{2}$$

First let's consider the problem when adjustment costs are linear. In that case adjustment to a steady state is instantaneous. If current level of capital is K_0 , the cost of increasing stock to K_∞ and then keeping it there (i.e, buying δK_∞ each unit of time) is $w[K_\infty - K_0 + \delta K_\infty/r]$, so the marginal cost of an additional unit of capital is just $w(1 + \delta/r)$. The marginal benefit of an extra unit of capital is π'/r . Setting marginal benefit equal to marginal cost gives $\pi' = w(r+\delta)$. This equation determines the equilibrium capital stock in the absence of adjustment costs. The dynamics in this problem were trivial, because it is optimal to jump immediately to a steady state.

2) Now consider control problem with convex adjustment costs, where it is optimal to approach steady state gradually. The control problem is

$$(1) \quad \max_I \int_0^{\infty} e^{-rt} [\pi(K) - C(I)] dt$$

$$\dot{K} = I - \delta K$$

δ = depreciation rate

$$H = \pi(K) - C(I) + \lambda(I - \delta K)$$

FOC

$$(2) \quad -C'(I) + \lambda = 0, \quad \text{or } \lambda = C'(I) \quad (\text{Tobin's "q"})$$

$$(3) \quad \dot{\lambda} = (r + \delta)\lambda - \pi'(K)$$

In steady state

$$(1') \quad I = \delta K$$

$$(3') \quad (r + \delta)\lambda = \pi'(K)$$

using (1') (2) and (3')

$$(4) \quad (r + \delta)C'(\delta K) = \pi'(K) \quad (\text{in S.S.})$$

Note from (4) that if $C' \equiv w$, a constant, we get the equilibrium condition that we obtained by equating marginal benefit and cost, above.

[Discuss estimation methods for this kind of problem: (Hansen and Sargent) linear-quadratic structure; (Epstein) dynamic duality; (Hansen and Singleton) method of moments]

2. Review basic fish problem

What happens if we take a control problem without adjustment costs and add these costs?
e.g., fishing problem

x = stock of fish, y = harvest
 $U(x,y)$ = profits or utility from harvest

growth equation

$$(1) \quad \dot{x} = F(x,y), \quad x_0 \text{ given}$$

e.g.,

$$(1') \quad F(x,y) = f(x) - y$$

"Usual Problem"

$$(P1) \quad \max \int_0^{\infty} e^{-rt} U(x,y) dt \quad \text{s.t. (1)}$$

Remember what solution to this problem looks like

$$(2) \quad \begin{aligned} H &= U(\cdot) + \lambda[f(\cdot) - y] \\ U_y - \lambda &= 0 \end{aligned}$$

(Recall interpretation of (2).)

$$(3) \quad \dot{\lambda} = [r - f'(x)]\lambda - U_x$$

use (2) to write $y = y(\lambda, x)$

$$(2') \quad \Rightarrow U_{yy} dy + U_{yx} dx - d\lambda = 0$$

$$\frac{dy}{dx} = - \frac{U_{yx}}{U_{yy}}, \quad \frac{dy}{d\lambda} = \frac{1}{U_{yy}} < 0$$

e.g. if $U = py - C(y, x)$, $-U_{yx} = C_{yx} < 0 \Rightarrow \frac{dy}{dx} > 0$

higher x , lower M.C. of harvest

We can linearize (1') and (3), using (2) to eliminate y to obtain

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} \approx A \begin{pmatrix} x \\ \lambda \end{pmatrix}$$

where A is a matrix of partial derivatives. Saddle point stability requires $|A| < 0$

(Remember thm for 1 state variable autonomous control problems: If the optimally controlled system converges to a SS, the SS is a saddle point. Convergence is monotonic for x , but not necessarily for control. See K&S Section II.9. Discuss intuition for monotonicity of state: Optimal y is function of x , so state dynamics can be written as $\dot{x} = f(x) - y(x) \equiv g(x)$, an autonomous differential equation. If $g(x) = 0$, motion stops, so $g(x)$ can not change signs.)

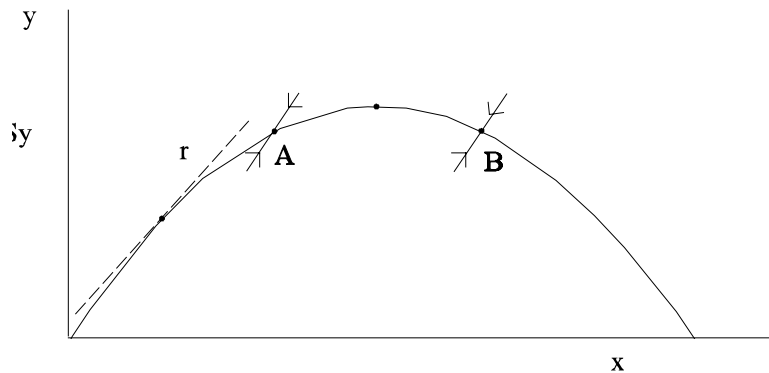
Using (2) and (3) we have at SS

$$[r - f'(x)]U_y = U_x$$

or

$$(4) \quad r - f'(x) = \frac{U_x}{U_y}$$

Note that if $U_x \equiv 0$, steady state solves $r = f(x)$, independent of function U .



Note that inclusion of x in payoff increases steady state $\frac{d[r - f'(x)]}{dx} = -f''(x) > 0$. (The

RHS of (4) equals 0 when U does not depend on x and is positive otherwise. The LHS is increasing in x for $f(x)$ concave.) Discuss economic intuition.

Mention that A and B stable only if feedback rule, $y(x)$, cuts $f(x)$ below. In other words, stability requires $dy/dx > df/dx$ at the steady state. Remember that I can find an expression for dy/dx just by manipulating the necessary conditions - I don't need to actually solve the problem.

3. Add adjustment costs to basic resource problem

(Here notes follow Feichtinger, et al. See also recent paper by Wirl in JEEM.)

What happens if we introduce costs of adjustment $\phi(a)$, so problem becomes

$$(P2) \quad \max \int_0^{\infty} e^{-rt} [U(x, y) - \phi(a)] dt$$

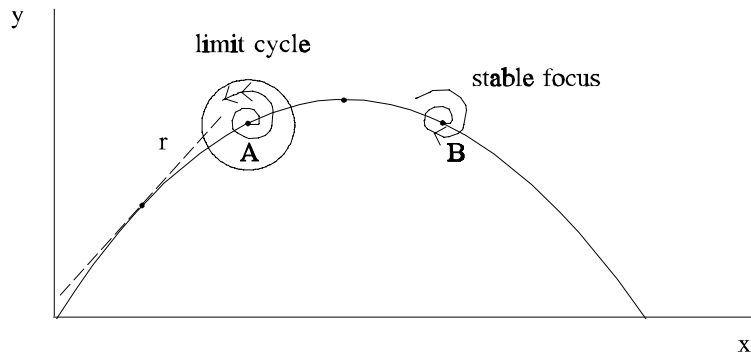
$$(1') \quad \dot{x} = f(x) - y \quad x_0 \text{ given}$$

$$(4) \quad \dot{y} = a \quad y_0 \text{ given}$$

(What if y_0 is free? How does problem change?)

Basic conclusion. If initial equil. is at a point like A ($f'(x) > 0$ and $r - f'(x) > 0$) introduction of costs of adjustment can lead to limit cycles. A point like B (where $f'(x) < 0$) can become a stable focus.

Possibilities with adjustment costs:



To characterize the steady states of (P2) we need to linearize a 4 dimensional system

$$H = U(x, y) - \phi(a) + \lambda[(x) - y] + \rho a$$

$$(5) \quad \frac{\partial H}{\partial a} = \rho - \phi'(a) = 0 \Rightarrow a = \gamma(\rho), \quad \gamma' > 0$$

$$(6) \quad \hat{\lambda} = \lambda(r - f'(x)) - U_x$$

$$(7) \quad \dot{\rho} = r\rho - U_y + \lambda$$

$$(8) \quad \begin{aligned} (4) \text{ and } (5) &\Rightarrow \\ \dot{y} &= \gamma(\rho) \end{aligned}$$

System is (1'), (6) - (8)

(Note that if we assume that adjustment costs are minimized when adjustment is 0, so that $\phi'(0) = 0$, then in steady state $\rho = 0$ (by (5)), so (7) implies that in steady state $U_y = \lambda$. Using this relation in (6), we see that the steady state(s) are unchanged by the introduction of adjustment costs. Adjustment costs change the dynamics around the steady state(s), but not the location of the steady state(s).

There is a formula (due to Dockner) for calculating eigenvalues of this system.

Write A as the Jacobian of the system. (It's denoted J in Fechtinger et al. paper) Let θ be a parameter in system, e.g. if $\phi(a) = \theta_0 a + \frac{\theta a^2}{2}$, θ is the obvious candidate (obvious because when $\theta = 0$, no adjustment costs.)

$$\dot{z} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\lambda} \\ \dot{\rho} \end{pmatrix} \approx AZ$$

Eigenvalues of A are a function of θ . (i) If \exists a value of θ , $\underline{\theta}$, such that there exist exactly two purely imaginary eigenvalues, $\varepsilon = \pm \omega i$, then $\underline{\theta}$ is critical value of bifurcation.

The pair of complex eigenvalues is $\varepsilon = D \pm \omega i$. (ii) You need $D(\underline{\theta}) = 0$ and $\frac{dD}{d\theta} \Big|_{\underline{\theta}} \neq 0$.

Conditions (i) and (ii) are relatively easy to check (or at least describe). If a third condition is also satisfied, then there exist limit cycles.