Larry Karp
Notes for Dynamics

X. Dynamic Games

Types of problems

1) "Symmetric" players. Each moves at the same time during a "period" (i.e., at each instant).

Ex. 1 Common property resource extraction.

x = stock, h_i = i's extraction policy

$$\dot{x} = f(x) - \sum_i h_i \qquad x(0) \text{ given}$$

Single "state variable", x. Do we have "natural boundary condition" as $t \to \infty$?

Is $f(x) \equiv 0$? Is U^i a function of h_j, $j \neq i$?

Ex. 2 Dynamic oligopoly with adjustment costs. K_j is stock of firm i's capital (e.g. physical plant, or advertising capital).

$\pi^i\,(K_i, K_j)$ is i's restricted profit function, $\phi(I^i)$ is adjustment costs.

$$\dot{K}_i = -\delta K_i + I_i$$

Here there are two state variables.

(2) "Asymmetric" (hierarchical) players.
(a) "Leader" moves before nonstrategic, but forward looking "follower".

Ex. 3 Optimal tariff for nonrenewable resource, competitive seller with rational expectations.

$$\dot{x} = -h \qquad x_0 \text{ given (resource stock)}$$

$$\dot{p} = r\big(p - c(x)\big) \qquad p_0 \text{ endogenous}$$

Equation for $\dot{p}$ results from seller's necessary condition for his optimization problem (Hotelling condition.)

Ex. 4 Optimal protection of dying sector with costly adjustment of labour

$\dot{L} = V(q,\tau)$ L_0 given
$\dot{q} = ???$ q_0 endogenous

τ is value of policy (wage subsidy, tariff) at a point in time. q is the value of being in the growing sector.

(b) Leader moves before strategic (forward looking) player

Ex. 5 Monopsonistic importer (leader) and monopolistic seller (follower) of nonrenewable resource.

$\dot{x} = -h$ x_0 given
$\dot{\lambda} = ???$ λ_0 endogenous

(c) Leader moves before 2 or more strategic agents who are playing a dynamic game.

Ex. 6 Dynamic Brander and Spencer. Take example 2 and let government choose a subsidy to influence investment decisions.

Types of Nash equilibria (strategies)

1) Open loop. Decisions are conditioned on time and initial value of state. Typically these are not subgame perfect. They may or may not be "dynamically consistent", or "time consistent"

2) Markov perfect (Feedback) Decisions are also conditioned on current state. These are subgame perfect.

2′) Differentiable, or "Smooth" M.P. Strategies are differentiable functions of state.

3) History dependent (Nonstationary) subgame perfect strategies. "Folk theorem" type results.

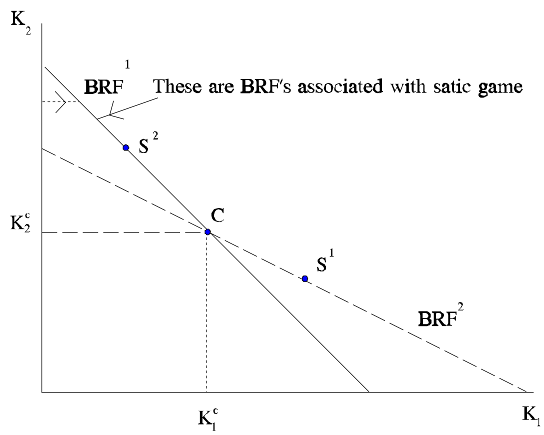
4) Closed loop. Strategies are state dependent, but need not be subgame perfect. (Empty threats OK).

Example that illustrates (1) and (2)

Spencer's capital accumulation game (follow Fudenberg and Tirole) illustrates difference between OLE and MPE, and nonuniqueness of MPE

irreversible investment $\dot{K}_i = a_i$ $0 \le a_i \le \bar{a}$, $\bar{a}$ is finite

K_i^C is static Cournot level.

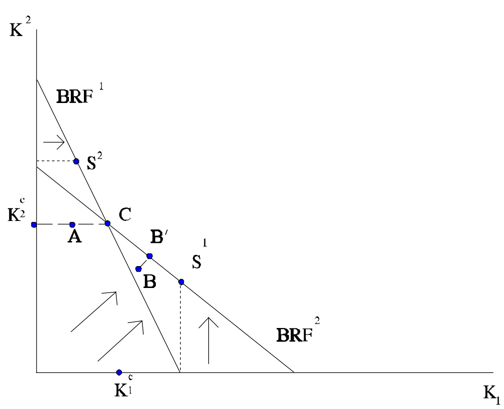


Firms maximize time average payoffs - only steady state matters.

Instantaneous payoffs $\pi_i(K_i, K_j)$

With open loop control (and low initial state) equil. steady state is *C*. Lots of equil. paths. (these don't matter).

MPE: condition investment on state.



On 45 degree arrows, both players choose same rate of investment, e.g. $\bar{a}$, Arrows show one MPE (equil. is defined by strategies) Choices defined at every state. These equilibria are piecewise differentiable functions of the state.

For this particular MPE, the upper envelope of BRF's equals the set of steady states. The particular steady state that emerges depends on initial condition.

S^i is Stackelberg equil. when *i* is leader.

Suppose initial condition is at A. Firm 2 can move steady state toward S^2.

Now explain why ∃ other, discontinuous MPE.

Point B is an equil if both firm prefer B to B' $\left(\pi^i(B) > \pi^i(B') \text{ for } i = 1,2\right)$. Restriction to MPE does not rule out "punishment strategies". These are discontinuous in state.

Mention finite horizon, discrete stage game: In last period a firm will invest if the state is below its BRF. In the second to last period, firms make investment decision knowing what will happen in last period. As horizon approaches infinity, equilibrium approaches the MPE identified by Spence.

Explain difference in OLE and MPE using 2 players, 1 state variable problem (e.g., example 1 - "Symmetric game") Consider matters from i's perspective. Agent j chooses control u_j (e.g. harvest from common property)

$$u_j^* = u_j(t, x_0) \quad \text{if OLE}$$

$$u_j^* = u_j(t, x) \quad \text{if MPE}$$

Different types of equil: open loop, feedback/Markov Perfect.

(1) $$\max \int_t^T L^i\left(x, u^i u^j, \tau\right) d\tau$$

$$\dot{x} = f\left(x, u^i u^j\right) \qquad x_t \text{ given}$$

$$H^i = L^i + \lambda^i f$$

(2) $$\frac{\partial H^i}{\partial u^i} = 0 \quad \text{(at interior sol'n)}$$

(3) $$\dot{\lambda}^i = -\frac{\partial H^i}{\partial x} = -\frac{\partial H^i}{\partial x} - \underline{\frac{\partial H^i}{\partial u_j}\frac{du_j}{dx}}$$

This term absent from OLE, present with MPE

Use of DPE to obtain MPE. (This is useful for stationary games, where $u_j^* = u_j(x)$. No time dependence, so look for stationary strategies.)

i's DPE is

(6) $$rV(x) = \max_{u_i} L\left(x, u_i\ u_j(x)\right) + V_x f\left(x, u_i, u_j(x)\right)$$

Note that I've substituted *j*'s equilibrium control rule into *i*'s decision problem. Thus, if I knew *j*'s decision rule, I could find *i*'s rule by solving a standard control problem. (Parallel with static game and states optimization problem.) Of course, I don't know *j*'s decision rule.

An example of use of DPE to obtain one MPE.

Fershtman and Kamien (like example 1)

Sticky prices. u_i is *i*'s rate of sales, $a - u_i - u_j$ = steady state price, $c + \frac{u_i}{2}$ = *i*'s average costs. s = speed of price adjustment. (Exercise: find OLE) Now find MPE. Firm i's value function is:

$$V^i(p) = \max_{u_i}\int_t^\infty \left(P - c - \frac{u_i}{2}\right)u_i e^{-r(\tau - t)}d\tau$$

$$\text{s.t.} \quad \dot{P} = s\left(a - u_i - u_j(P) - P\right)$$

(apply eqn (6))

(7) $$rV^i(P) = \max_{u_i}\left[\left(P - c - \frac{u_i}{2}\right) + V_P^i s\left(a - u_i - u_j(P) - P\right)\right]$$

"Guess" that $u_j(P) = \alpha + \beta P$ (linear control rule)

and that $V^i(P)$ is quadratic: $V^i = \gamma_0 + \gamma_1 P + \frac{\gamma_2 P^2}{2}$

(Note: in the game, unlike the control problem, this guess is not innocuous. The guess implies a particular MPE. We return to this point below.

(1) Sub this guess into (7)
(2) Perform maximization ⇒ linear control rule for i
If agents i and j are symmetric, then the control rule for i is also $u_i = \alpha + \beta P$, with α and β depending on: α, β, γ_0, γ_1, γ_2.

(3) Sub equilibrium u_i into (7) and equate coefficient of 1, x, x^2 to find γ_0, γ_1, γ_2.

Characteristics of equil. as $s \to \infty$ (so problem "resembles" static problem.)

OLE S.S. → steady state of static duopoly
MPS S.S. → lower price than static duopoly.

Firms have a "preemptive incentive". By selling more today a firm lowers future price, and lowers rival's future sales. ($\beta > 0$).

Reynolds (like example 2)

i's problem

$$V^i(K_i, K_j) = \max_{u_i} \int_0^\infty e^{-rt}\left[(a - K_i - K_j)K_i - \frac{cu_i^2}{2}\right]dt$$

s.t. $\dot{K}_i = u_i$
$\dot{K}_j = u_j$

again, the DPE is

(8) $$rV^i = \max_{u_i}\left[(a - K_i - K_j)K_i - \frac{cu_i^2}{2} + V^i_{K_i}u_i + V^i_{K_j}u_j\right]$$

"Guess" linear control of j

(9) $$u_j = \theta + \beta K_j + \rho K_i$$

and linear-quadratic value function:

(10) $$V^i = \alpha_0 + \alpha_1 K_i + \alpha_2 K_j + \left(\alpha_3 K_i^2 + \alpha_4 K_j^2 + 2\alpha_5 K_i K_j\right)$$

Recipe:

(1) Sub (9) and (10) into (8)

(2) max w.r.t. u_i to obtain $u_i = \theta + \beta K_i + \rho K_j$ (for a symmetric game.)

(3) Sub this control rule into (8) and equate coefficients of 1, K_i, K_j, K_i^2, K_j^2, K_iK_j to obtain equations for α_s $s = 0,1,...5$, and θ, β, ρ.

Exercise: Find steady state for OLE, show that it is the same as equil of corresponding static game.

Qualitative features of MPE S.S.

as $c \to 0$ MPE SS remains above static equil.
as $c \to \infty$ MPE SS approaches static equil.

Explanation based on pre-emptive incentive ($\rho < 0$)

Explanation for nonuniqueness of MPE (Tsutsui and Mino). Show how to obtain ODE for symmetric equil in single state variable problem.

P = stock of pollution, E_i = i's emission

(11) $$\dot{P} = f(P) + \Sigma E_j$$

i's flow payoff $L(\overset{-}{P}, \overset{+}{E_i})$

Review control problem $\left(\dot{P} = f(P) + E\right)$

$$H = L(P,E) + \lambda\left(\delta(P) + E\right)$$

(12) $$\frac{\partial L}{\partial E} + \lambda = 0 \Rightarrow E = E(p,\lambda)$$

(13) $$\dot{\lambda} = \lambda\left(r - f'(p)\right) - \frac{\partial L}{\partial P}$$

Diff (12) w.r.t. time, use (11) and (12) and (13)

$$\frac{\partial^2 L}{\partial E^2}\dot{E} + \frac{\partial^2 L}{\partial E \partial p}\left(f(P) + E\right) + \left(-\frac{\partial L}{\partial E}\right)\left(r - f'(P)\right) - \frac{\partial L}{\partial E}$$

$$\Rightarrow$$

(14) $$\dot{E} = h\left(E,P\right)$$

Divide (14) by (11)

(15) $$\frac{\dot{E}}{\dot{P}} = \frac{dE}{dP} = \frac{h(E,P)}{f(P) + E} = y(P,E)$$

(15) is an ODE

B.C. is obtained by solving

(16) $$h(E,P) = 0 = f(P) + E$$

Two equations in two unknowns. Alternatively, could set $\dot{\lambda}$, and $\dot{P}$ equal to 0:

(16a) $$\dot{\lambda} = \lambda\big(r - f'(P)\big) - \frac{\partial L\big(P, E(\lambda,P)\big)}{\partial P} = 0 = f(P) + E(\quad)$$

Why doesn't this work for dynamic game? Let $n = 2$, look for a symmetric equilibrium. $E_j = g(P)$ is rival's decision rule. i's Hamiltonian is

$$H = L(P,E) + \lambda\big(f(P) + g(P) + E\big)$$

(12′) $$\frac{\partial L}{\partial E} + \lambda = 0 \Rightarrow E = E(P,\lambda)$$

(13′) $$\dot{\lambda} = \lambda\big(r - f'(P)\big) - \frac{\partial L}{\partial P} - \lambda g'(P)$$

We can still obtain an ODE like (15). For a symmetric equil, $E \equiv g(P)$, so aggregate emissions $\equiv G(P) = 2g(P)$, or $E = \frac{G(P)}{2}$, and $g'(P) \equiv \frac{G'(P)}{2}$. Eqn (11) is now

(11′) $$\dot{P} = f(P) + G(P)$$

Replace E and g in (12′) and (13′) by $\frac{G}{2}$. Differentiate (12′) w.r.t. time, using (11′) – (13′) to obtain

(14′) $$\dot{G} = H(G,P)$$

Divide (14′) by (11′) to obtain

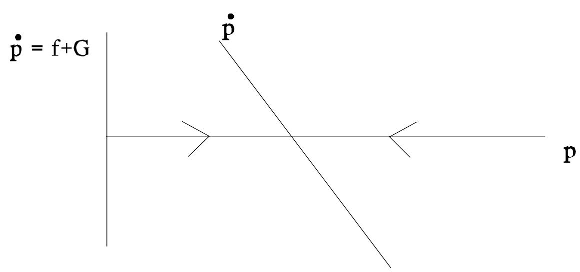
(15′) $$\frac{\dot{G}}{\dot{P}} = \frac{dG}{dP} = \frac{H}{\big(f(P) + G\big)} \equiv Y(P,G)$$

The problem is: What is the B.C.? (11′) and (13′) imply (in S.S.)

$$(16) \qquad f(P) + G(P,\lambda) = 0 = \lambda\big(r - f'(P)\big) - \frac{\partial L\big(E(P,\lambda),P\big)}{\partial P} - \lambda g'(P)\Big|_{P^*}$$

(16) is 2 equations in 3 unknowns, P^*, λ^*, and $-g'(P)\big|_{P^*}$.

Stability implies some restrictions on steady state.

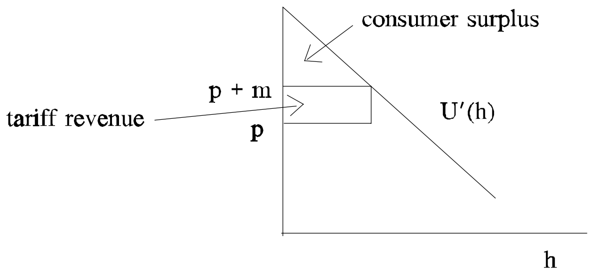


Stability requires $\frac{d\dot{P}}{dP} < 0$, or $f' + G' < 0$, evaluated at a steady state.

This still leads to a family of MPE (candidates). Each member of this family solves the same ODE (15′) but with a different B.C.

Example of asymmetric game. Competitive seller of nonrenewable resources, monopsonistic importer chooses tariff or import growth.

p = price seller receives, $p + m$ = domestic tariff inclusive price for buyers. m = tariff



(1) $\dot{x} = -h$ stock constraint

(2) $\dot{P} = r\left(P - c(x)\right)$ Hotelling condition

Buyers flow of welfare is $U(h) - Ph$ = consumer surplus + tariff revenue

OLE (explain why this is dynamically inconsistent.)

$$H = U(h) - Ph - \rho h + \lambda r\left(P - c(x)\right)$$

ρ = shadow value of stock
λ = shadow value of price
P^d = domestic tariff inclusive price $\equiv U'(h)$

FOC's

(3) $$m \equiv U'(h) - P = \rho$$

(4) $$\dot{\lambda} - r\lambda = h - r\lambda \Rightarrow \dot{\lambda} = h$$

use $\lambda_0 = 0$ (why?)

(4′) $$\lambda(t) = \int_0^t h\,d\tau = x_0 - x_t > 0$$ discuss

(5) $$\dot{\rho} = r\rho + \lambda c'(x)$$

(3) – (5) $\Rightarrow$ optimal trajectory satisfies

(6) $$U''(h)\dot{h} = r\left[U'(h) - c(x)\right] + \left(x_0 - x\right)c'(x)$$

OLE is dynamically inconsistent unless $c'(x) \equiv 0$

MPE. Look for a function $P(x)$ such that when importer solves

$$\int_0^\infty e^{-rt}\left[U(h) - P(x)h\right]dt$$

subject to (1), the solution satisfies (2)

e.g. $U'(h) = \alpha - h,\ c(x) = (\alpha - bx)$

Buyer's DPE is

(6) $$rJ(x) = \max_h\left[U(h) - P(x)h\right] - J_x h$$

"guess" $J(x) = \frac{\gamma x^2}{2},\ P(x) = \alpha - \beta x$

Sub guess in to (6). Perform max to obtain

(7) $$h = (\beta - \gamma)x$$

Sub (7) into (6) to obtain

$$\frac{r\gamma x^2}{2} = (\beta - \gamma)^2 \frac{x^2}{2}$$

(8) $$r\gamma = (\beta = \gamma)^2$$

Now use equil. condition (2)

(9) $$\dot{P} = r(P - C) = r\left[(\alpha - \beta x) - (\alpha + bx)\right] = r(b - \beta)x$$

By definition

(10) $$\dot{P} = \left(\frac{dP}{dx}\right)\dot{x} = \beta(\beta - \gamma)S$$

use guess for $P(x)$ and (7)

Equate (9) and (10)

(11) $$r(b - \beta) = \beta(\beta - \gamma)$$

(8) and (11) are two equations in two unknowns, β and γ. Can show ∃ unique sol'n with positive β.