

## I. Basic Ideas of ODE's

- 1) Basic terms of ordinary differential equations (ODE's).
- 2) Basics of phase plane analysis.
- 3) Solutions and stability of linear ODE's.
- 4) Linear approximations of nonlinear ODE's.

What is a differential equation? ODE

$$\dot{x} \equiv \frac{dx}{dt} = f(x, t)$$

solution to ODE  $x_t = \phi(t)$  family of curves

$\dot{x} = f(x)$  autonomous (no dependence on  $t$ , or more generally, on independent variable)

e.g.  $\dot{x} = \alpha x$

$$\frac{dx}{dt}/x = \alpha$$

$$\frac{dx}{x} = \alpha dt$$

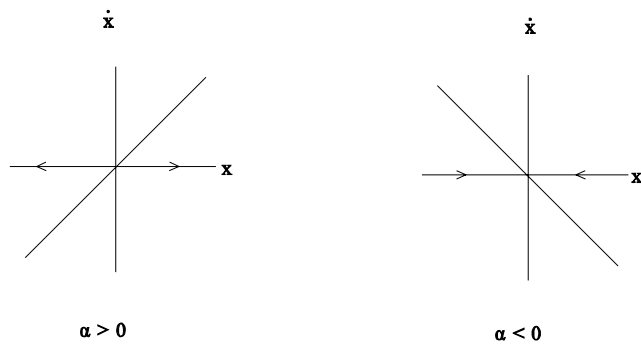
$$\ln(x) = \alpha t + c$$

$$x = ce^{\alpha t}$$

$$x = \varphi(t)$$

$$\text{or } x = \varphi(t; c)$$

$$\text{or } x = \varphi(t; c_0, t_0)$$



Suppose I have a boundary condition (or initial condition)

$$x_3 = 7$$

$$7 = ce^{\alpha 3} \Rightarrow c = 7e^{-3\alpha}$$

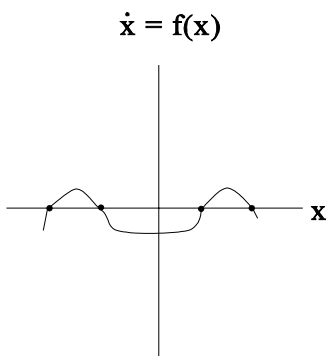
$$x_t = 7e^{\alpha(t-3)}$$

B.C.  
↓ ↓

$$x_t = \phi(t; t_0, x_0)$$

Definitions: (Equilibrium point) steady state  $x^*$   
 $\dot{x} = 0 = f(x^*, t)$

Linear autonomous example, steady state is 0.



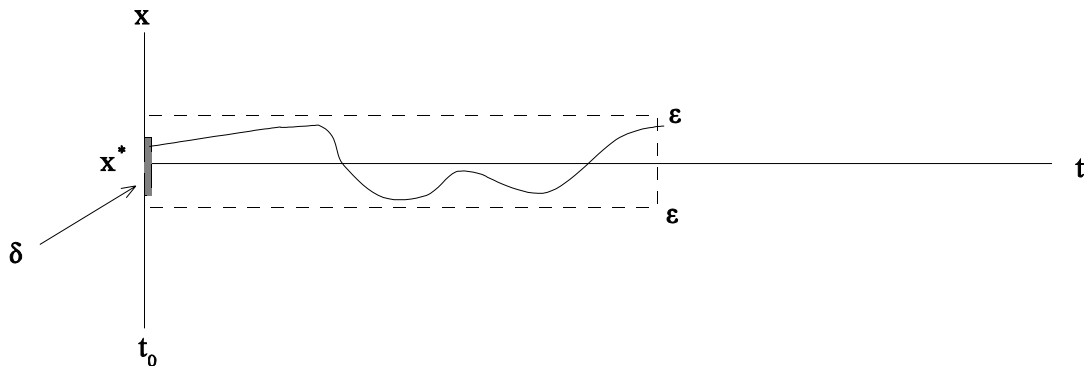
In general case, can have multiple steady states.

Stability:  $x^*$  is stable if  $\forall \epsilon > 0, t_0 \geq 0$

$$\exists \delta(\epsilon, t_0) \text{ such that } |x_0 - x^*| < \delta$$

$$\Rightarrow |\phi(t; t_0, x_0) - x^*| < \epsilon$$

(If I begin close to steady state, I stay close.)



Asymptotic Stability: requires  $x^*$  to be stable and  $\phi(t; t_0, x_0) \rightarrow x^*$  as  $t \rightarrow \infty$   $\square$

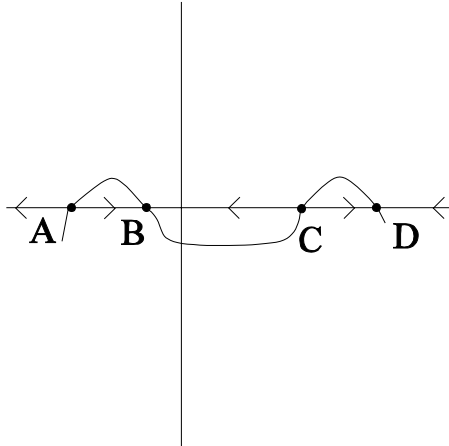
(If I begin close to S.S, I approach it.)

Global asymptotic stability (GAS).  $x$  converges to  $x^*$  regardless of initial condition.

Note: for linear example  $\dot{x} = \alpha x$ ,  $x^* = 0$  is GAS iff  $\alpha < 0$ . We can see this by looking at solution, or graph.

Consider nonlinear autonomous example with multiple steady states  $\dot{x} = f(x)$ . Some steady states are stable, other are not stable.

In the following figure, B & D are asymptotically stable. The stable points B & D are separated by the unstable steady state C. Note: At a steady state,  $f(x) = 0$  and at a stable steady state,  $f(x)$  is decreasing. Therefore, if  $f(x)$  is continuous, there must be an unstable steady state between any two stable steady states.



Analysis of 2 dimensional systems.

Suppose we are given a autonomous system.

$$\dot{x} = F(x,y) \quad \dot{y} = G(x,y)$$

We want to know what happens to  $x$  and  $y$  over time. (Example: fish stocks that interact, state and control in optimal control problems, capital stock and shadow value of capital in rational expectation models.)

See Clark page 169 for following existence theorem:

Theorem: If  $F$  &  $G$  are continuous, bounded, with continuous first derivatives, then  $\exists$  unique solution to system that satisfies the initial condition  $x(0) = x_0, y(0) = y_0$ . Solution is either defined

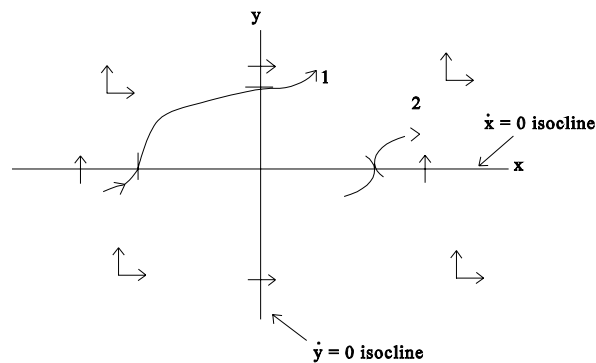
for all  $t \geq 0$ , or norm approaches  $\rightarrow \infty$  Norm:  $\sqrt{x_{(t)}^2 + y_{(t)}^2}$

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Example of analysis using phase space

$$\dot{x} = y^2 \quad \dot{y} = x^2$$

Suppose that I had an initial condition for this system (e.g, the values of  $x$  and  $y$  at time  $t=0$ ) and that I solved the system of equations to obtain the value of  $x$  and  $y$  for arbitrary  $t$ . That is, I find the functions  $x(t)$  and  $y(t)$  that satisfy the system of ODEs and the initial condition. I can then project these functions on to the  $x,y$  plane. The resulting curve in  $x,y$  space is called a trajectory. (It is more precise to call it the projection of a trajectory onto phase space, but I will often refer to it merely as a "trajectory"). For different initial conditions I have different trajectories. The sketch of these different trajectories is called a phase portrait. We will analyze the evolution (over time) of the variable  $x$  and  $y$  by studying the phase portrait.



The first step in drawing the phase portrait is to find two special isoclines. In general, an isocline is a curve defined by the set of points where the slope of the trajectory ( $dx/dy$ , or  $dy/dx$ ) is constant. For the purpose of constructing phase portraits, it is convenient to use the isoclines where the slope of a trajectory is either 0 or  $\infty$ .

$$y = 0 \text{ (horizontal axis)} \Rightarrow \dot{x} = 0$$

$$x = 0 \text{ (vertical axis)} \Rightarrow \dot{y} = 0$$

slopes of trajectory are 0 or  $\pm \infty$  on those isoclines. These two isoclines divide phase space into regions known as isosectors. In this example, the isoclines are straight lines, so there are four isosectors. More generally, the isoclines might intersect more than once, resulting in more than four isosectors.

Note that the figures above include arrows; drawing these arrows is the second step in constructing the phase portraits. These arrows tell you the direction of movement of the solutions  $x(t)$  and  $y(t)$  as  $t$  increases. Remember that the trajectories graphed in  $x,y$  space are the projections of the paths, in 3 dimensions, of  $t$ ,  $x(t)$  and  $y(t)$ .

Here's how we find the directional arrows. In each isosector we want to know the direction of change of  $x$  and  $y$ . (There are 4 isosectors in this example.) On the horizontal axis,  $dx/dt=0$ , and at any point off the axis,  $x$  is increasing. Therefore  $x$  is increasing at any point off the axis. Draw arrows going to the right in each isosector. Similarly, draw arrows going up (to indicate that  $y$  is increasing) in each isosector. The direction of a trajectory at any point in phase space is "between" the directional arrows you have just drawn, i.e it is Northeast.

In this example, the trajectories point Northeast in all isosectors, so this is rather a trivial case. More generally, the directions of the trajectories might be (and typically are) different in different isosectors.

The curves labeled 1 & 2 are examples of trajectories.

Slope of trajectories given by

$$\frac{dy}{dx} = \frac{G(x)}{F(y)} = \frac{x^2}{y^2}$$

For this example we can solve

$$\frac{dy}{dx} = \frac{x^2}{y^2} \qquad y^2 dy = x^2 dx$$

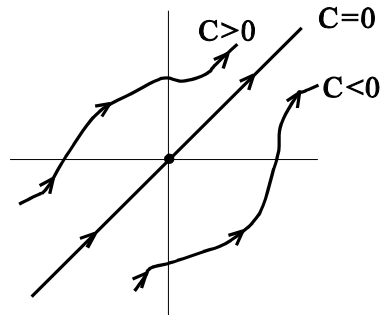
$$\int y^2 dy = \int x^2 dx$$

$$\frac{y^3}{3} = \frac{x^3}{3} + \tilde{C}$$

$$y = (x^3 + C)^{1/3}$$

If I had an initial condition, I could eliminate C. In most cases we cannot obtain an explicit solution for the trajectories, but we can often learn something about the dynamic behavior by using graphical methods.

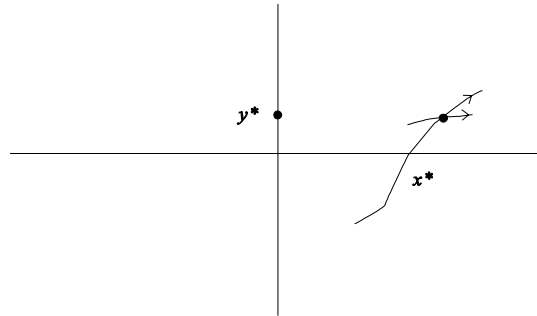
The graph below shows three trajectories for three values of C. When C equals 0, the trajectory is a straight line. Note that all of the arrows are pointing Northeast, so on any trajectory x and y increase over time. The only trajectory that goes through the steady state corresponds to C=0, i.e. it is the straight line. It is obvious from the explicit solution,  $y = (x^3 + C)^{1/3}$ , that this straight line has slope = 1. If the initial condition is on this straight line and is negative (i.e. x=y<0) the values of x(t) and y(t) approach 0 as t increases. If the initial condition is off this half-line, the values of x(t) and y(t) approach infinity as t increases.



Steady state occurs at the intersection of the isoclines (which were obtained by setting the time derivatives equal to 0). In this example, the steady state is  $x = y = 0$ .

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Theorem: Trajectories never cross.



The figure shows a case where the trajectories do cross. To see (intuitively) why this outcome is impossible, note that at  $x^*, y^*$ , trajectories 1 and 2 have different slopes, but both are given by

$\frac{G(\cdot)}{F(\cdot)}$ . This is a contradiction, so the hypothesis that the trajectories cross must be false.

At a steady state, where both  $G(\cdot)$  and  $F(\cdot)$  are equal to zero (i.e. at a steady state), the ratio  $\frac{G(\cdot)}{F(\cdot)}$  is

not defined. Depending on the nature of the steady state, there may be infinitely many trajectories either converging to or diverging from a steady state (as with a node), or there may be one (a focus) or none (a center). But we haven't discussed these terms yet.]

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Predator prey models with limited growth. Use this to illustrate (1) the type of info you can get from phase plane analysis and (2) the use of linear approximation around a steady state.

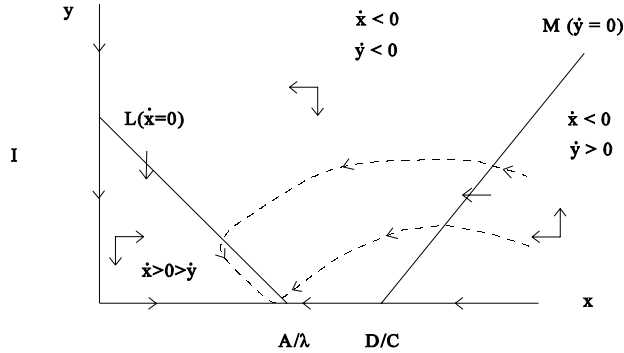
$$\begin{aligned} \dot{x} &= (A - By - \lambda x) x && \text{prey} \\ \dot{y} &= (Cx - D - \mu y) y && \text{predator.} \end{aligned}$$

All parameter +. more predator ( $y$ )  $\Rightarrow$  further decline of prey ( $x$ ). More prey ( $x$ )  $\Rightarrow$  growth of predator.  $\lambda$  is a measure of "congestion".

If  $\lambda = \mu = 0 \Rightarrow$  Volterra Lotke eqn.

$\lambda > 0 \Rightarrow$  a maximum prey stock size, when  $y = 0$

$\mu > 0 \Rightarrow$  competition amongst predator for prey.



The figure shows the case where  $D/C > A/\lambda$ . Exercise: give economic interpretation of this inequality; e.g., if  $C$  is small, presence of prey does not increase growth of predators by much, so predator eventually dies out; or, if  $\lambda$  is large, there is a lot of "congestion" in prey population, preventing it from growing large enough to support predator.

On curve L,  $\dot{x} = 0$

$$L: A - By - \lambda x = 0$$

On curve M,  $\dot{y} = 0$

$$M: Cx - D - \mu y = 0$$

Curves L & M are isoclines

Begin analysis by finding isoclines

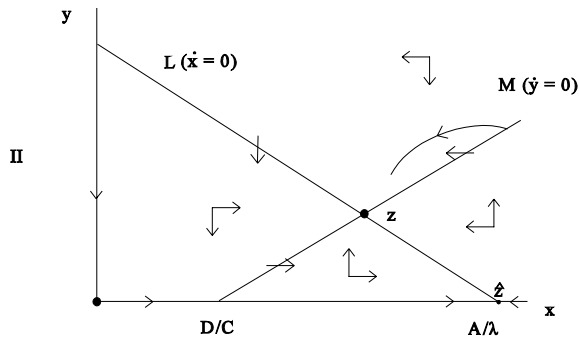
Explain how to find directional arrows.

$$\left. \frac{\partial \dot{x}}{\partial y} \right|_L = -Bx < 0, \text{ so above } L, \dot{x} < 0$$

On the curve L,  $\dot{x} = 0$ . The above inequality says that if I start out on the curve L and increase  $y$  by a small amount (move North on the graph), then  $\dot{x}$  decreases. Since  $\dot{x} = 0$  on L, it must therefore be the case that  $\dot{x} < 0$  above the curve. That is, on trajectories above the curve L,  $x$  is getting smaller (as  $t$ , time, increases). We represent this movement by a leftward arrow, as shown.

Isoclines: a curve in phase space along which slope of trajectory is constant. For example, any trajectory that passes through the curve on which one variable ( $x$  or  $y$ ) is not changing (the curves L or M in the above example) has a slope of 0 or infinity on that curve.

Dotted lines in I show two possible trajectories. If initial value of  $x > 0$ , system converges to  $(A/\lambda, 0)$ . Predators die out in first case.



This figure shows the case  $D/C < A/\lambda$

How do we determine what kind of equilibrium  $z$  and  $\hat{z}$  are?

Basic idea: Find a linear approximation to nonlinear autonomous system,

$$(1) \quad \begin{aligned} \frac{dX}{dt} &= f(X, Y) \\ \frac{dY}{dt} &= g(X, Y) \end{aligned}$$

at an equilibrium point  $X_0, Y_0: f(X_0, Y_0) = g(X_0, Y_0) = 0$

Suppose that  $f$  and  $g$  analytic in the neighborhood of the steady state. (For our purposes, think of “analytic” as meaning “sufficiently smooth that the Taylor approximation converges to the true value of the function. See Judd, page 196, who defines analytic in terms of a polynomial on the complex plane.) If a function  $f$  is analytic in some set other than a particular point  $y$ , then  $y$  is known as a singularity of  $f$ , or a singular point. For example, the function  $f=1/x$  is analytic for all  $x$  other than 0.  $X=0$  is a singular point.)

$$\dot{X} = f(X, Y) = \overset{= 0}{f(X_0, Y_0)} + \overset{= a}{\frac{\partial f(X_0, Y_0)}{\partial X}}(X - X_0) + \overset{= b}{\frac{\partial f(X_0, Y_0)}{\partial Y}}(Y - Y_0) + o(X - X_0, Y - Y_0)$$

A term is  $o(\Delta)$  means  $\lim_{\Delta \rightarrow 0} \frac{o(\Delta)}{\Delta} = 0$

Similarly for  $Y$ . (Remember,  $\Delta$  is the Euclidean norm of  $(X - Y_0, Y - Y_0)$ . I defined this norm above.)

$$\dot{Y} = g(X, Y) = c(X - X_0) + d(Y - Y_0) + o(X - X_0, Y - Y_0)$$

$$c = \frac{\partial g(X_0, Y_0)}{\partial X} \quad d = \frac{\partial g(X_0, Y_0)}{\partial Y}$$

$$(2) \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + o(x - X_0, y - Y_0) \quad \begin{matrix} x = X - X_0 \\ y = Y - Y_0 \end{matrix}$$

We may have to check that remainder term really is  $o(x - x_0, y - y_0)$

Stability Theorem in the First Approximation:

- (i) If 0 soln of (2) is asymptotically stable, then  $X_0, Y_0$  of (1) is asymptotically stable.
- (ii) If 0 soln of (2) is unstable then  $X_0, Y_0$  is an unstable solution of (1).

Three possibilities:

- |                                 |               |                        |
|---------------------------------|---------------|------------------------|
| 0 soln of (2) is                | $\Rightarrow$ | $(x_0, y_0)$ of (1) is |
| i) asymptotically stable        | $\Rightarrow$ | asym. stable           |
| ii) unstable                    | $\Rightarrow$ | unstable               |
| iii) stable but not asym stable | $\Rightarrow$ | ???                    |

In order to know how to analyze (1), we need to be able to analyze (2). In general we want to

analyze a linear system. I'm going to begin by transforming the system so the steady state is the origin. If I begin with  $\dot{\xi} = A\xi + b$ , define  $z = \xi + A^{-1}b$ . Note that  $\dot{\xi} = 0$  when  $\xi = -A^{-1}b$ , i.e., when  $z = 0$ .  $z=0$  is the steady state of the transformed system. (I'm assuming that  $A^{-1}$  exists. The set of steady states has the same dimension as the kernel of  $A$ , so if  $A$  is singular, the steady state is not unique.)

$$\begin{matrix} \dot{z} & = & A & z \\ nx1 & & nxn & \end{matrix}$$

Bottom line: stability of system depends on characteristic roots of  $A$

We know how to solve  $\dot{z} = a_1 z$ :  $z_t = z_0 e^{a_1 t}$

Try to make original system look like this. If  $A$  was diagonal, it would be easy, e.g.

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$\lambda_i$  is characteristic root,  $P_i$  is characteristic vector

$$A P_i = \lambda_i P_i$$

if  $(A - \lambda I)P = 0$  for  $P \neq 0$

$$\Rightarrow |A - \lambda I| = 0$$

characteristic equations

$$\begin{matrix} \overbrace{A} & \overbrace{P} & = & \overbrace{A} & \overbrace{P} \\ \underbrace{A} & \underbrace{[P_1, P_2, \dots, P_n]} & = & \underbrace{A} & \underbrace{[P_1, P_2, \dots, P_n]} \\ nxn & nxn & & nxn & nxn \end{matrix} \begin{matrix} \overbrace{\Lambda} \\ \left[ \begin{matrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{matrix} \right] \\ \underbrace{\Lambda} \\ nxn \end{matrix}$$

$$AP = P\Lambda$$

$$\text{if } P^{-1} \exists \Rightarrow P^{-1}AP = \Lambda$$

There are several possible cases, depending on whether there exist real or complex eigenvalues, and on whether the matrix  $P$  is nonsingular. The following notes are not exhaustive. See a text on ODE's, such as Boyce and DiPrima.

Suppose  $P$  is nonsingular

$$\begin{aligned} w &\equiv P^{-1}z \Rightarrow \dot{w} = P^{-1}\dot{z} \\ z &= Pw \\ \dot{z} &= Az \\ P\dot{w} &= APw \Rightarrow \dot{w} = P^{-1}APw = \Lambda w, w_i = \lambda_i w_i \end{aligned}$$

Now suppose in addition that the roots are real

$$w_i(t) = w_i(0)e^{\lambda_i t}$$

$$\begin{matrix} w(t) = e^{\Lambda t} & w(0) \\ nx1 & nxn & n > 1 \end{matrix}$$

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$$

This equality hold only when the matrix in the exponent is a diagonal

matrix. For a general, i.e. non-diagonal matrix  $A$ ,

$$\therefore e^{At} \neq \begin{bmatrix} e^{a_{11}t} & \dots & e^{a_{1n}t} \\ \vdots & \ddots & \vdots \\ & & e^{a_{nn}t} \end{bmatrix}$$

use

$$w(t) = e^{\Lambda t} w(0)$$

$$w = P^{-1}z, \text{ so } w(0) = P^{-1}z(0)$$

$$P^{-1}z(t) = e^{\Lambda t} P^{-1}z(0)$$

$$z(t) = P e^{\Lambda t} c$$

$$\text{where } w(0) \equiv c = P^{-1}z(0) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

$$(3) \quad z(t) = c_1 P_1 e^{\lambda_1 t} + c_2 P_2 e^{\lambda_2 t} + \dots + c_n P_n e^{\lambda_n t}$$

Conclude:  $A$  is stable iff  $\lambda_i \leq 0$   
 $A$  is asym. stable iff  $\lambda_i < 0$   
 $A$  is unstable if some  $\lambda_i > 0$

Suppose some roots are imaginary:  $\lambda_i = a \pm bi$

e.g. 
$$\dot{z} = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} z$$

$$0 = \begin{vmatrix} 1-\lambda & -1 \\ 5 & -3-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 2 \Rightarrow$$

$$\lambda = \frac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm i \quad i = \sqrt{-1}$$

Example for  $n$  dimensional case. Suppose  $\lambda_1 = \lambda + \mu i$ ,  $\lambda_2 = \lambda - \mu i$ ,  $\lambda_3 \dots \lambda_n$  are real.

Can show eigenvectors  $P_1 = a + bi$ ,  $P_2 = a - bi$ , where  $a$  and  $b$  are vectors. (Eigenvectors associated with complex eigenvalues are complex.)

$$(4) \quad z(t) = c_1 \theta(t) + c_2 \rho(t) + c_3 P_3 e^{\lambda_3 t} + \dots + c_n P_n e^{\lambda_n t}$$

$$\theta(t) = e^{\lambda t} (a \cos \mu t - b \sin \mu t)$$

$$\rho(t) = e^{\lambda t} (a \cos \mu t + b \sin \mu t)$$

Point: stability depends on real part of complex eigenvalue.

Suppose  $\exists$  a set of linearly independent eigenvectors, i.e.,  $P$  is singular.

e.g. 
$$\dot{z} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} z$$

$$0 = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} \Rightarrow \lambda_1 = \lambda_2 = 2$$

To find  $P$ , solve 
$$\begin{pmatrix} 1-2 & -1 \\ 1 & 3-2 \end{pmatrix} \begin{pmatrix} p_{11} \\ p_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} -p_{11} - p_{12} &= 0 \\ p_{11} + p_{12} &= 0 \end{aligned} \Rightarrow \exists \text{ a single eigenvector which is proportional to } \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(see Boyce and Diprima page 294 for example of repeated eigenvalue with linearly independent eigenvectors.)

If eigenvalue is of multiplicity 2, and only 1 linearly independent eigenvector,  $P_1$ , the solution to ODE is of form

$$(5) \quad z_t = c_0 P_1 e^{\lambda t} + c_1 P_1 t e^{\lambda t} + c_2 \eta e^{\lambda t}$$

↑  
see B& D for details on calculation

Note that for all cases stability depends on sign of (real part of)  $\lambda_i$

A system may converge for some I.C. in region of steady state. The set of initial conditions (a line or hyperplane) from which the system converges is called the stable manifold.

Suppose  $\lambda_1, \lambda_2, \dots, \lambda_m < 0, \lambda_{m+1}, \dots, \lambda_n \geq 0^*$

If I set  $c_{m+1} \dots c_n = 0$ , (3) becomes

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\*In this case, equilibrium is a saddlepoint

$$(3') \quad z(t) = c_1 P_1 e^{\lambda_1 t} + \dots + c_m P_m e^{\lambda_m t} + 0 \dots + 0$$

This converges,  $\Rightarrow$

$$z_0 = \begin{bmatrix} P_1 & P_2 & \dots & P_m & P_{m+1} & \dots & P_n \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

recall  $c \equiv P^{-1}z(0)$

In other words, the system converges if initial condition is a linear combination of the eigenvectors associated with stable eigenvalues.

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A saddlepoint is an equilibrium which is approached from some region, not from others. (A saddlepoint has some + and some - roots.)

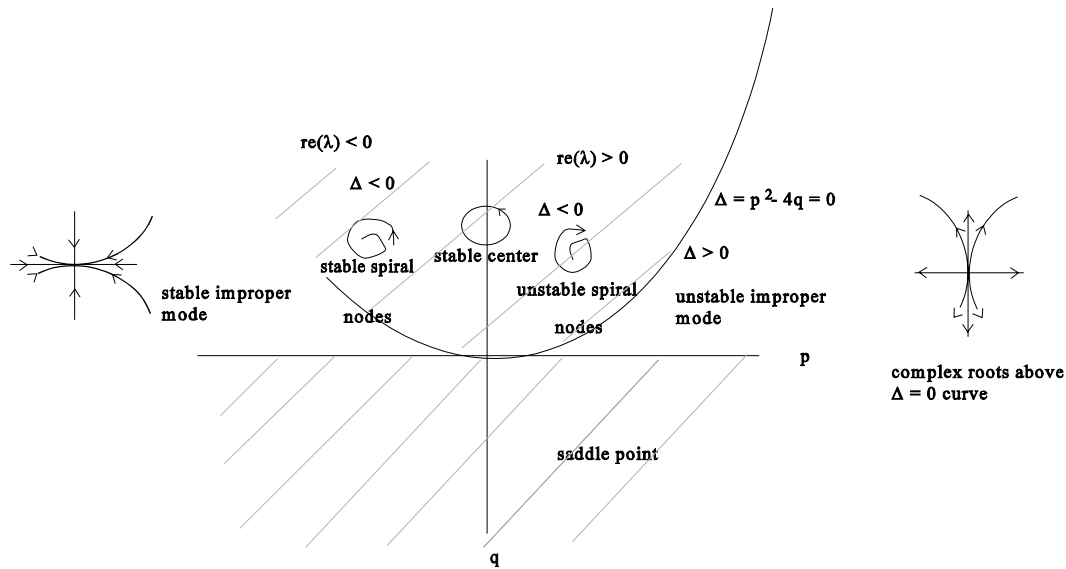
Some useful facts (which hold for a general matrix A -- not just a two-dimensional matrix)

$$: \quad \text{trace } A = \sum \lambda_i \quad |A| = \prod \lambda_i$$

Pg 409 Boyce and Diprima

Two dimensional systems

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \begin{aligned} p &\equiv a + d = \text{trace} \\ q &\equiv ad - bc = |A| \end{aligned}$$



our rules  $\Rightarrow \lambda_1 + \lambda_2 = p$   
 $\lambda_1 \lambda_2 = q$

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - p\lambda + q \quad \lambda = \frac{p}{2} \pm \frac{\sqrt{p^2 - 4q}}{2}$$

$p = \text{trace}(A)$ ,  $q = \text{determinant}(A)$   
 $\Delta \equiv p^2 - 4q$ . If (the discriminant)  $\Delta < 0$ , roots are complex

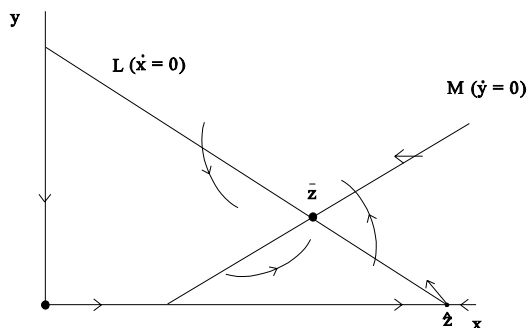
Summary: We can determine local stability by examining roots of linear system. For a two dimensional system we can determine the nature of the steady state by using the graph above, and calculating the trace, determinant and the discriminant.

Return to predator-prey example.

prey  $\dot{x} = (A - By - \lambda x) x \equiv f(x,y)$   
 predator  $\dot{y} = (cx - D - \mu y) y \equiv g(x,y)$

$$L: A - By - \lambda x = 0$$

$$M: Cx - D - \mu y = 0$$



		$\bar{z}$	$\hat{z}$
$f_x$	$= A - By - 2\lambda x$	$-\lambda \bar{x}^{**}$	$A - 2\lambda \hat{x}$
$f_y$	$-Bx$	$-B\bar{x}$	$-B\hat{x}$
$g_x$	$Cy$	$C\bar{y}$	$0$
$g_y$	$Cx - D - 2\mu y$	$-\mu \bar{y}$	$C\hat{x} - D$

at  $\bar{z}$

$$z' = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \approx \begin{bmatrix} -\lambda \bar{x} & -B\bar{x} \\ C\bar{y} & -\mu \bar{y} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$p = -\lambda \bar{x} - \mu \bar{y} < 0, \quad q = \lambda \mu \bar{x} \bar{y} + B C \bar{x} \bar{y} > 0$$

$p < 0$ ,  $q > 0 \Rightarrow$  stable (If the trace is negative, at least one root must be negative. Since the determinant is positive, both roots have the same sign. Therefore both roots must be negative.)  
Conclude that  $\bar{z}$  is asymptotically stable.

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\*\* (To obtain this expression, use  $A - B\bar{y} - \lambda \bar{x} = 0$ )

$$\text{At } \hat{z} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \approx \begin{bmatrix} A-2\lambda\hat{x} & -B\hat{x} \\ 0 & c\hat{x}-D \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(1,1) term =  $-\lambda \hat{x} = -A < 0$  (use  $A - \lambda \hat{x} = 0$ )

(2,2) term =  $\frac{CA}{\lambda} - D > 0$  (since  $\hat{z}$  is below  $M$ )

$$q = (-) \cdot (+) < 0 \Rightarrow \text{saddlepoint}$$

Any point that starts on  $x$  axis converges to  $\hat{z}$ . The  $x$  axis is stable manifold.

An example where linearization does not tell us whether nonlinear system is stable is Voleterra-Lotka:  $\lambda = \mu = 0$  (no upper bound on equilibrium., e.g. if  $y = 0$ , prey grows without bound.)

Can show by linearizing system around steady state that  $\exists$  (pure) imaginary eigenvalues (real part = 0). In this case, linear system is stable, but not asymptotically stable, so analysis of linear system does not tell us whether steady state of nonlinear system is a center or a spiral (focus). However, this system can be solved explicitly (See Clark).

Saddlepoint equilibria are important in optimal control problems. Our standard problem involves one "state variable", such as a stock of fish, where the initial condition is given. The necessary conditions for the control problem gives us an ODE for another variable. (You will see in Section 3 that for the simple control problem we have a choice of analyzing several two-dimensional systems, all of which give the same information, but look slightly different. For example, the second variable can be either the shadow value of the state, or it can be the optimal control, such as harvest).

In a broad class of autonomous problems, the optimally controlled state variable converges to a steady state. In this case, the other variable (the shadow value, or the control, depending on how we analyze the problem) also converges to a steady state. I assume here that it is optimal for the state to approach a limiting value -- a steady state.

The *initial* value of the state variable (e.g. the stock of fish) is given exogenously (in most problems). We have to pick the initial value of the other variable (the shadow value, or control) in order to drive the system to a steady state.

Here I want to anticipate future discussions a bit, and suggest why it makes sense that steady states are (often) saddle points. Consider the alternatives.

Suppose that the steady state was an unstable node (all trajectories in the neighborhood of the steady state diverge from it. In that case, the optimally controlled system cannot reach the optimal steady state -- obviously a nonsensical conclusion.

Suppose that the steady state is stable but not asymptotically stable (that is, a center). In that

case, the optimal trajectory circles around the steady state, so for a range of values of the state, there are many different optimal control (or shadow values) -- another nonsensical conclusion.

Suppose that the steady state is a focus, so that the trajectory circles toward the steady state (i.e., it is asymptotically stable). Again, we would have the nonsensical result that for a range of values of the state there is more than one optimal level of the control. (If there is more than one state variable, things are more complicated. The steady state in that case can be a focus.)

Suppose that the steady state is a stable node. In that case, the system would converge regardless of the initial value of the endogenous variable (either the shadow value of the state or the control variable). In that case, any value of the control variable would satisfy the necessary condition -- i.e. the necessary condition would be useless in telling us the optimal value of the control.)

The only remaining possibility is for the steady state to be a saddlepoint. That means that the system will converge only if we pick the initial control correctly.

(I do not to leave you with the impression that all two-dimensional systems worth studying have steady states that are saddle points. There is a large literature in macroeconomics on the indeterminacy of competitive equilibria. In a class of these models, there is a unique steady state, but that steady state is a stable node. In this case, there are uncountably many rational expectations competitive equilibria. This situation typically can arise in a variety of situations, including in models where there is sufficiently strong non-convexities.)

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For a non-linear system, there may be multiple steady states. The following discussion refers to a particular steady state.

For two-dimensional autonomous systems, there are two trajectories through a saddlepoint. Each of these trajectories is called a separatrix. The arrows on *one* separatrix point towards the saddlepoint; for initial conditions on this separatrix, the system approaches the steady state. This trajectory is called the stable trajectory (sometime "stable saddle-path" or the "stable arm" or the stable manifold). If the initial condition is not on the stable trajectory, the trajectory moves away from the saddlepoint.

The two trajectories through the saddlepoint -- the separatrices -- divide phase space into regions<sup>\*\*\*</sup>. The dynamic motion in these different regions is qualitatively different -- hence the term "separatrix".. (We have already noted that trajectories do not cross each other, so no trajectories cross a separatrix.)

For two-dimensional linear systems with saddle points, the separatrices are straight lines; all other trajectories in this case are curves. How do we know that the separatrix is a straight line in the two-dimensional system? Because we know that if we have a saddle point (in a two dimensional system)

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<sup>\*\*\*</sup>I mentioned above that the isoclines obtained by setting time derivatives equal to 0 also divide phase space into regions, known as iso-sectors. These iso-sectors are not the same as the regions that are bounded by separatrices. Remember that the isoclines are sets of points where trajectories have the same slope -- zero or infinity for the isoclines that I have used. Trajectories do cross isoclines. Trajectories do not cross other trajectories -- including separatrices.

there is one stable and one unstable eigenvalue, and one eigenvector associated with each of these eigenvalues. The stable manifold -- i.e. the set of initial conditions from which trajectories converge to the steady state -- is a linear combination of eigenvectors associated with the stable eigenvalues. In this case, though, there is only one of these vectors. Thus, the stable manifold is a straight line.

This fact gives us a simple method of calculating the separatrices for two dimensional linear systems. Rather than finding the eigenvalues and then solving for the eigenvectors, we can obtain the slope of the eigenvectors directly, by solving a quadratic equation. If we define variable in such a way that the origin is a steady state (as I did above), knowing the slope tells me the eigenvector.

Here is the basic idea. Begin with the linear system  $\dot{x} = ax + by$ ,  $\dot{y} = cx + dy$ , where the origin is the steady state. Suppose that this steady state is a saddlepoint. (This is easy to check using the trace and determinant of the system.) We know that each saddlepath is a straight line, i.e. it solves  $y=ex$ , for some number  $e$ , that is,  $dy/dx = e$  (on a separatrix). We also know that on the saddle path the dynamics are described by the original system, so  $\dot{y}/\dot{x} = dy/dx$ . Thus,  $\dot{y}/\dot{x} = e$  on the separatrix. On the separatrix, we can rewrite the original system as

$$\dot{x} = ax + bex, \dot{y} = cx + dex$$

Use this system to eliminate  $y$  from the expression  $\dot{y}/\dot{x} = e$ . The result is

$$(cx+dex)/(ax+bex)=e \quad \text{or} \quad e(ax+bex)=(cx+dex) \quad \text{or} \quad (a+bee-c-de)x=0$$

Since this equation must hold for all  $x$ , it must be the case that

$$(a+bee-c-de)=0$$

This equation is quadratic in  $e$ . The solution gives two values of  $e$ , the slopes of the eigenvalues.

One reason for being interested in the stable manifold -- a line in the two-dimensional system -- is that it provides the basis for obtaining a numerical approximation of the optimal control rule in a one-state autonomous control problem. You will be asked to do this in one of the problem sets

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Two examples of analysis of two-dimensional dynamic systems:

1) Brander, James and Scott Taylor "The Simple Economics of Easter Island: a Ricardo-Malthus Model of Renewable Resource Use". AER 1998, pp 119 - 138.

They use a simple general equilibrium, renewable resource model to describe the rise and fall of an isolated civilization, like the one on Easter Island. The economy produces two goods and has Ricardian technology. "The Environment" (a stock variable) affects the marginal productivity of

labor in one sector. The other state variable is the stock of human population. An increase in the environment encourages growth of human population (it becomes easier to get food). An increase in human population degrades the environment, since extraction increases. Authors use the predator-prey model described in these notes, with the environment as "the prey" and mankind as "the predator". Analysis proceeds by linearizing dynamics around the interior steady state and checking whether the roots are real or imaginary. They show that a stable interior equilibrium can be either an "improper node" (real roots) with monotonic adjustment or a spiral node (imaginary roots), with trajectories spiralling into the steady state. Spirals occur if the intrinsic growth rate of the environment is sufficiently small. Using "reasonable" parameter values, they show that it is possible to get a trajectory in which human population increases for a time, then crashes, eventually reaching a low level.

2) Krugman, P. (1991) "History versus Expectation" QJE 651 - 67. (See also the 1993 QJE article by Benabu and Fukao which corrects a technical error in Krugman's analysis. See Section 2 of my paper Fundamentals Versus Beliefs under Almost Common Knowledge at <http://are.Berkeley.EDU/~karp/> for a two-period version of this model.)

Two sector economy, one sector has increasing returns to scale. Hold output prices fixed (e.g., small open economy), consider allocation of labor between sectors. Let  $L$  be the amount of labor in IRTS sector,  $1 =$  total amount of labor, so  $1 - L$  is amount of labor in CRTS sector. The VMP = wage in IRTS sector is  $a + bL$ ,  $b > 0 \Rightarrow$  ICTS. The VMP of labor in CRTS sector  $\equiv c$ .  $a, b, c$  are constants.

Suppose that cost of moving from between sectors is  $\gamma[(L)^2]/2$  where  $\gamma > 0$ . The marginal cost of moving is therefore the absolute value of  $\gamma L$ . An individual that migrates at time  $t$  pays the price equal to marginal cost. (You can think of "migration services" being competitively supplied.) In this model there are convex adjustment costs: the marginal cost of moving increases with the speed at which movement takes place.

The wage differential at a point in time is  $m(L) = a - c + bL$ . Assume  $c > a$ ,  $a+b > c$ . If all labor is in IRTS sector, wage there is higher. If no labor is in IRTS sector, wage there is lower.

Suppose, for example, that an individual migrates from the CRTS sector to the IRTS sector at time  $t$ . We have seen that this individual pays the price (the absolute value of)  $\gamma L$ . How much does the individual gain? Define  $T$  as the time at which all migration ceases. ( $T$  may be infinite, but in this problem it happens to be finite.) An instant after  $T$ , an individual can migrate without paying any costs, since by definition no one else is migrating at that time, so  $L = 0 = \gamma L$  (= the price of migration). By migrating at  $t$  rather than  $T$ , the individual gains  $q$ , defined as

$$q = \int_0^T e^{-rt} m(t) dt \quad \Rightarrow \quad \dot{q} = rq - m .$$

The variable  $q$  obeys  $\dot{q} = rq - m$ . Equation of this sort appear in many contexts. Interpret this fundamental "no arbitrage condition": Dividend (the wage differential) plus capital gain ( $\dot{q}$ ) = opportunity cost of holding the stock ( $rq$ ).

If an individual worker takes all prices (the cost of moving and the wage differential at every point in the future) as given and has rational expectations, equilibrium requires

$$\gamma \dot{L} = q$$

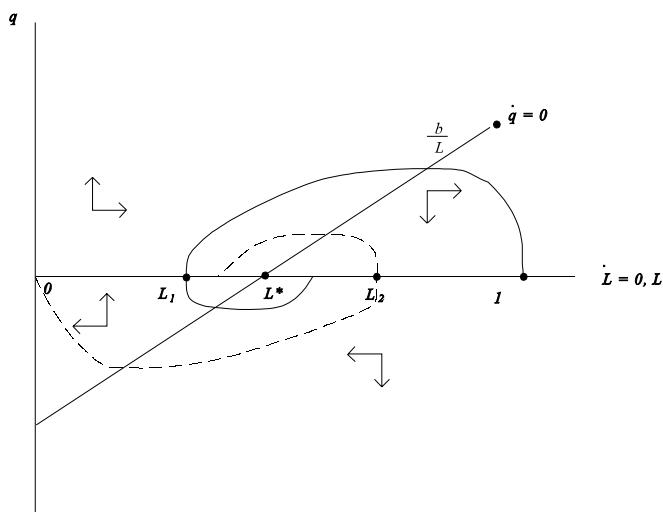
The equilibrium condition simply says that an individual is willing to pay "what it is worth" to get into a different sector.

We can write the equilibrium conditions for this model as

$$\begin{pmatrix} \dot{q} \\ \dot{L} \end{pmatrix} = \begin{bmatrix} r & -b \\ 1/\gamma & 0 \end{bmatrix} \begin{pmatrix} q \\ L \end{pmatrix}$$

The trace =  $r > 0$  and the determinant =  $b/\gamma > 0$ , and the discriminant is  $\Delta = r^2 - 4b/\gamma$ . Since both the trace and the determinant are positive, we know that the interior steady state,  $L^*$ , is unstable. If  $\Delta < 0$  (i.e. if  $r$  or  $\gamma$  are small, or if  $b$  is large) the steady state is an unstable spiral; if  $\Delta > 0$  the steady state is an unstable node.

Figures (13) and (14) show the two possible phase portraits of these two differential equations. In



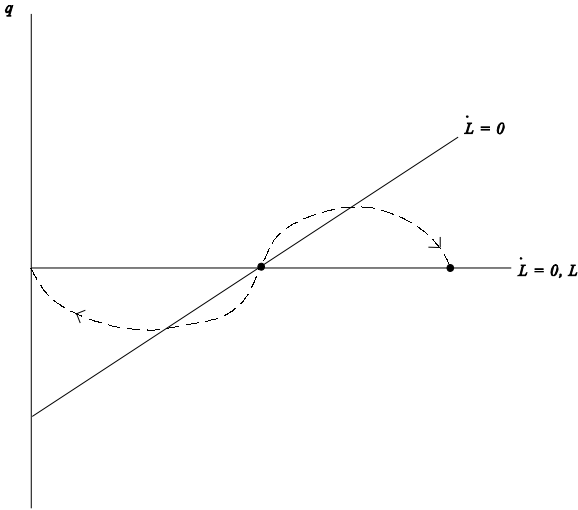
**Figure 12** Phase space of  $L, q$  (expectations matter)

both cases, there are three steady states, the boundaries  $L = 0$  and  $L = 1$ , and the interior equilibrium denoted  $L^*$ . We saw that the interior steady state is unstable. Instability occurs because of IRTS; See section 2 of my paper for the intuition.

$L^*$  could be an unstable spiral, as shown in figure 13 or an unstable node as shown in figure 14.

Expectations matter: The dashed trajectory (figure 13) leads to  $L = 0$ , the solid trajectory leads to  $L = 1$ .

For initial conditions  $\in (L_1, L_2)$  economy could be on either trajectory, so "expectations matter" (i.e., expectations determine trajectory of economy). For initial conditions less than  $L_1$  or greater than  $L_2$  there is only one equilibrium trajectory, so here "history matters. In other words, for the parameters that lead to figure 13, whether expectations matter depend on the initial condition.



Only history matters: If  $L^*$  is an unstable node (figure 14), only history matters. For initial conditions  $L < L^*$ , economy moves to low steady state, for initial conditions  $L > L^*$ , economy moves to high steady state.

Property of unstable equilibrium depends on parameters (which determine the value of  $\Delta$ ). Unstable node if and only if  $r$  is large relative to  $b$  and  $\gamma$ .

**Figure 13** Phase space of L,q (history matters)