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Notes for Dynamics,
Fall, 1998

I. Basic Ideas of ODE's

1) Basic terms of ordinary differential equations (ODE's).
2) Basics of phase plane analysis.
3) Solutions and stability of linear ODE's.
4) Linear approximations of nonlinear ODE's.

What is a differential equation? ODE

$$\dot{x} \equiv \frac{dx}{dt} = f(x,t)$$

solution to ODE $x_t = \phi(t)$ family of curves

$\dot{x} = f(x)$ autonomous (no dependence on t, or more generally, on independent variable)

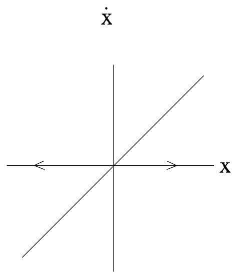
e.g. $\dot{x} = \alpha x$

$$\frac{dx}{dt}/x = \alpha$$

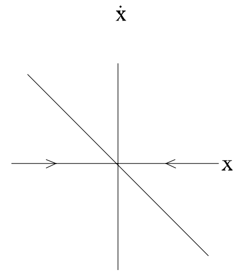
$$\frac{dx}{x} = \alpha dt$$

$$\ln(x) = \alpha \mathrm{t} + c$$

$$x = ce^{\alpha t}$$
$$x = \phi(t)$$
$$\text{or } x = \phi(t;c)$$
$$\text{or } x = \phi(t;c_0,t_0)$$



$\alpha > 0$



$\alpha < 0$

Suppose I have a <u>boundary condition</u> (or initial condition)

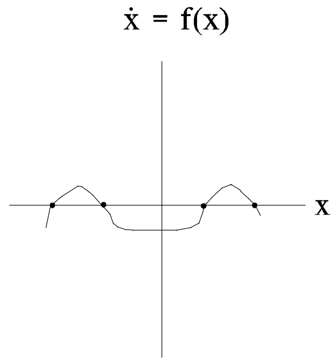
$$x_3 = 7$$
$$7 = ce^{\alpha 3} \Rightarrow c = 7e^{-3\alpha}$$

$$x_t = 7e^{\alpha(t-3)} \qquad\qquad x_t = \phi(t; \overset{\text{B.C.}}{\overset{\downarrow\ \downarrow}{t_0, x_0}})$$

Definitions: (Equilibrium point) steady state x^*

$$\dot{x} = 0 = f(x^*, t)$$

Linear autonomous example, steady state is 0.



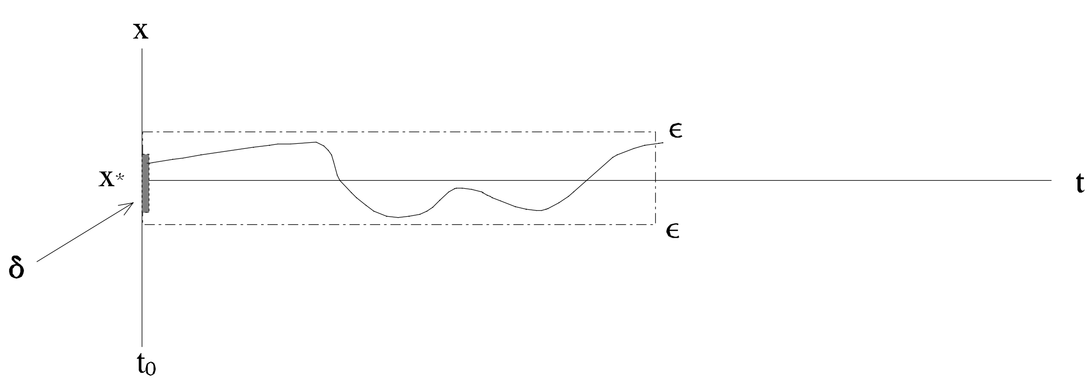
In general case, can have mutliple steady states.

<u>Stability</u>: x^* is stable if $\forall \varepsilon > 0$, $t_0 \geq 0$

$$\exists \quad \delta\,(\varepsilon, t_0) \text{ such that } \quad |x_o - x^*| < \delta$$

$$\Rightarrow \quad |\phi(t; t_0, x_0) - x^*| < \varepsilon$$

(If I begin close to steady state, I stay close)



Asymptotic Stability: requires x^* to be stable and $\phi(t;\, t_0,\, x_0) \to x^*$ as $t \to \infty$

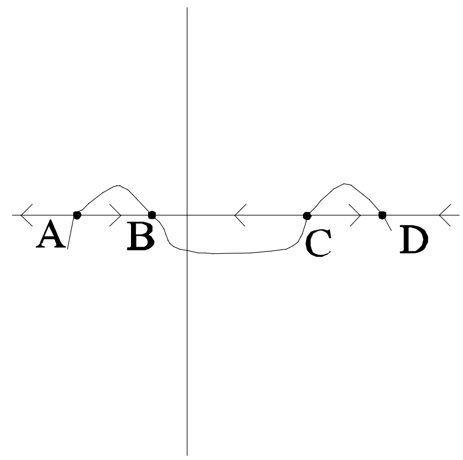
(If I begin close to S.S, I approach it.)

Global assumptions stability (GAS). x converges to x^* regardless of initial condition.

Note: for linear example $\dot{x} = \alpha x$, $x^* = 0$ is GAS iff $\alpha < 0$. We can see this by looking at solution, or graph.

For nonlinear autonomous example with multiple steady states $\dot{x} = f(x)$

B & D are asym. stable (Stable points B & D separated by unstable point C). Note: At a steady state, f(x) = 0 and at a stable steady state, f(x) is decreasing. Therefore, if f'(x) is continuous, there must be an unstalbe steady state between any two stable steady states.



Analysis of 2 dimensional systems.

Suppose we are given a <u>autonomous</u> system.

$$\dot{x} = F(x,y) \qquad \dot{y} = G(x,y)$$

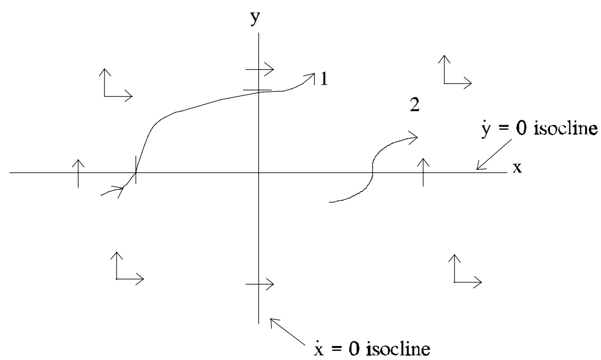
We want to know what happens to x and y over time. (Example: fish stocks that interact, state and control in optimal control problems, capital stock and shadow value of capital in rational expectation models.)

Clark page 169

Thrmn. F&G are continuous, bounded, with continuous first derivatives ⇒ ∃ unique solution to system that satisfies I.C. $x(0) = x_0$, $y(0) = y_0$. Solution is either defined for all $t \geq 0$, or norm approaches → ∞ Norm: $\sqrt{x_{(t)}^2 + y_{(t)}^2}$

Example of analysis using phase space

$$\dot{x} = y^2 \qquad \dot{y} = x^2$$



isoclines: curves defined by the set of points where slope of trajectory (dx/dy, or dy/dx) is constant[1]

$y = 0$ (horizonal axis) $\Rightarrow \dot{x} = 0$
$x = 0$ (veritcal axis) $\Rightarrow \dot{y} = 0$

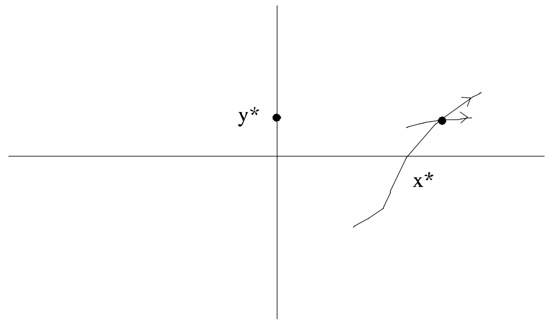
slopes of trajectory are 0 or $\pm\,\infty$ on those isoclines

1 & 2 are examples of trajectories.

Slope of trajectories given by $\dfrac{dy}{dx} = \dfrac{G(\ )}{F(\ )} = \dfrac{x^2}{y^2}$

Thrm: Trajectories never cross

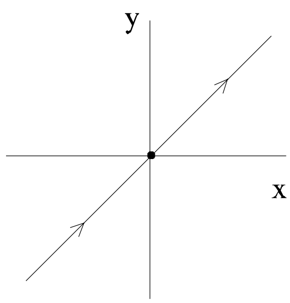
[1] For the purpose of constructing phase potraits, we are usually interested in the isoclines where the slope of a trajectory is either 0 or ∞.



This outcome is impossible. At x^*, y^*, trajectories 1 and 2 have different slopes, but both are given by $\dfrac{G(\ )}{F(\ )}$ $\Rightarrow\Leftarrow$.

Steady state obtained where isoclines cross - in this example, at $x = y = 0$.

Every point in phase space is part of some trajectory, so there is a trajectory through the origin



For this example we can solve

$$\frac{dy}{dx} = \frac{x^2}{y^2} \qquad\qquad y^2 dy = x^2 dx$$

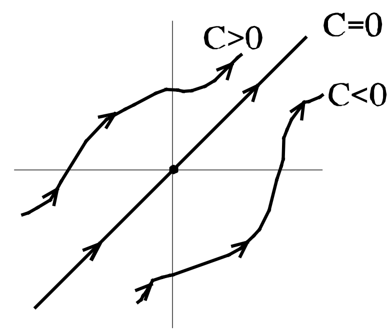
$$\int y^2 dy = x^2 dx$$

$$\frac{y^3}{3} = \frac{x^3}{3} + \tilde{C}$$

$$y = (x^3 + C)^{1/3}$$

If I had an I.C., I could eliminate C

Separatrix: a line in phase space that trajectories never cross; on a separatrix, the slope of the trajectory is the same as the slope of the line. (In other words, the separatrix is a trajectory. If a trajectory starts on one side of a separatrix, it never crosses to the other side. If a trajectory starts on a separatrix, it remain on it.) Trajectory though C = 0 is a separatrix in example above [2].



How to find the sepatrix (or seperatrices). Use $dy/dx = \dot{y}/\dot{x}$. On a separtrix, $dy/dx = K$, a constant. This equation implies that the sepatrix must be given by $y = k + Kx$, where k is also a constant. Use $y = k + Kx$ to eliminate y in the equation $\dot{x}/\dot{y} = K$. If we do this for the example above we obtain the equation $K(k + 2kKx + K^2x^2) = x^2$. This relation must hold for an interval of x, i.e. for all x in the domain of the separtrix - not just at a point. This fact implies that $k = 0$ and $K^3 = 1$, or $K = 1$. (This method of finding the unknown parameters k and K is sometimes referred to as "equating coefficients of terms of the same order", i.e. the terms that involve the constant, x, and x^2. You will encounter this method again when we study the "linear quadratic control problem. Problem set 1 asks you to use this method to calculate the separatrices for a specific example.)

For linear system, ∃ two separatrices. This is important in control problems. As an exercise, find the two separatrices for two-dimensional (autonomous) linear system $\dot{x} = ax + by$, $\dot{y} = cx + dy$, using the procedure described above. (Use (i) $\dot{y}/\dot{x} = dy/dx =$ (on a separtrix) K and (ii) $y = k + Kx$. Now you should find that K is the solution to a quadratic; thus there are two sepatrices.)

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Predator prey models with limited growth. Use this to illustrate (1) the type of info you can get from phase plane analysis and (2) the use of linear approximation around a steady state.

$$\dot{x} = (A - By - \lambda x)\,x \qquad \text{prey}$$
$$\dot{y} = (Cx - D - \mu y)\,y \qquad \text{predator.}$$

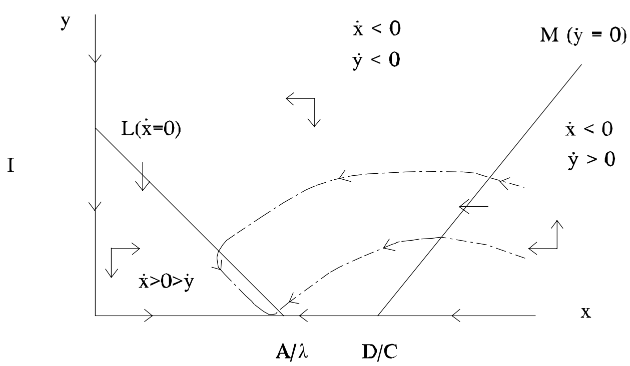
All parameter +. more predator (y) ⇒ further decline of prey (x). More prey (x) ⇒ growth of predator.

[2] Note the difference between a separatrix and an isocline. An isocline is a curve (possibly but not necessarily a straight line), and an isocline need not be (typically, is not) a trajectory. Rather, all trajectories that cross an isocline have the same slope.

If $\lambda = \mu = 0 \Rightarrow$ Volterra Lotke eqn.

$\lambda > 0 \Rightarrow$ a maximum prey stock size, when $y = 0$

$\mu > 0 \Rightarrow$ competition amongst predator for prey.



One possibility $D/C > A/\lambda$
Exercise: give economic interpretation. e.g., if c is small, presence of prey does not increase growth of predators by much, so predator eventually dies out; or, if λ is large, there is a lot of "congestion" in prey population, preventing it from growing large enough to support predator.

On curve L, $\dot{x} = 0$ L: $A - By - \lambda x = 0$

On curve M, $\dot{y} = 0$ M: $Cx - D - \mu y = 0$

Curves L & M are isoclines

Begin analysis by finding isoclines

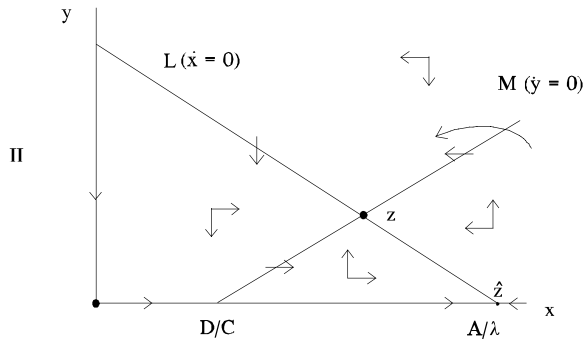
Explain how to find directional arrows.

$$\frac{\partial \dot{x}}{\partial y}\bigg|_L = -Bx < 0 \text{ , so above } L,\ \dot{x} < 0$$

isoclines: a curve in phase space along which slope of trajectory is constant. (For e.g. one variable is constant.)

Dotted lines in I show two possible trajectories. If initial value of x > 0, system converges to (A/λ, 0) or to (0, 0).[3] Predators die out in first case.

[3] If initial condition is close to y axis, *above the line L*, then trajectory can hit the y axix and thereafter approach the origen.



Another possibility D/C < A/λ

How do we determine what kind of equilibrium z and $\hat{z}$ are?

Basic idea: Find a linear approximation to nonlinear autonomous system,

$$\text{(1)} \qquad \frac{dX}{dt} = f(X,Y) \qquad \frac{dY}{dt} = g(X,Y)$$

at an equilibrium point X_0, Y_0: $f(X_0\ Y_0) = g(X_0\ Y_0) = 0$

f and g analytic (smooth)

$$\dot{X} = f(X,Y) = \overbrace{f(X_0 Y_0)}^{=\,0} + \overbrace{\frac{\partial f(X_0,Y_0)}{\partial X}}^{=\,a}(X - X_0) + \overbrace{\frac{\partial f(X_0,Y_0)}{\partial Y}}^{=\,b}(Y - Y_0) + o(X - X_0, Y - Y_0)$$

A term is $o(\Delta)$ means $\lim\limits_{\Delta \to 0} \dfrac{o(\Delta)}{\Delta} = 0$

Similarly for Y (Δ is the norm of (X-Y_0, Y-Y_0))

$$\dot{Y} = g(X,Y) = c(X - X_0) + d(Y - Y_0) + o(X - X_0, Y - Y_0)$$

$$c = \frac{\partial g(X_0 Y_0)}{\partial X} \quad d = \frac{\partial g(X_0 Y_0)}{\partial Y}$$

(2) $$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + o(X - X_0, Y - Y_0) \qquad \begin{aligned} x &= X - X_0 \\ y &= Y - Y_0 \end{aligned}$$

We may have to check that remainder term really is $o\,(x - x_0,\, y - y_0)$

Stability Theorem in the First Approximation:

(i) If 0 soln of (2) is <u>asymptotically</u> stable, then X_0, Y_0 of (1) is asymptotically stable.
(ii) If 0 soln of (2) is <u>unstable</u> then $X_0 Y_0$ is an unstable solution of (1).

Three possibilities:

0 soln of (2) is	$\Rightarrow$	$(x_0\ y_0)$ of (1) is
i) asymptotically stable	$\Rightarrow$	asym. stable
ii) unstable	$\Rightarrow$	unstable
iii) stable but not asym stable	$\Rightarrow$	???

In order to know how to analyze (1), we need to be able to analyze (2). In general we want to analyze a linear system. I'm going to begin by transforming the system so the steady state is the origen. If I begin with $\dot{\xi} = A\xi + b$, define $z = \xi - A^{-1}b$. Note that $\dot{\xi} = 0$ when $\xi = A^{-1}b$, i.e., when z = 0. z=0 is the steady state of the transformed system. (I'm assuming that A^{-1} exists. The set of steady states has the same dimension as the kernal of A, so if A is singular, the steady state is not unique.)

$$\underset{nx1}{\dot{z}} = \underset{nxn}{A}\ z$$

Bottom line: stability of system depends on characteristic roots of A

We know how to solve $\dot{z}_1 = a_1 y$: $\quad z_t = z_0 e^{a_1 t}$

Try to make original system look like this. If A was diagonal, it would be easy, e.g.

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

characteristic root characteristic vector

$$A\, P_i = \underset{\text{characteristic root}}{\lambda_i}\, \underset{\text{characteristic vector}}{P_i}$$

if $(A - \lambda I)P = 0$ for $P \neq 0$

$\Rightarrow |A - \lambda I| = 0$

characteristic equations

$$\underset{n x n}{A}\,\underset{n x n}{\overbrace{\left[P_1, P_2, \ldots, P_n\right]}^{P}} = \underset{n x n}{\left[P_1, P_2, \ldots, P_n\right]}\underset{n x n}{\overbrace{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}}^{\Lambda}}$$

$$AP = P\Lambda$$

$$\text{if } P^{-1} \exists \Rightarrow P^{-1}AP = \Lambda$$

Suppose P is nonsingular

$$w \equiv P^{-1}z \Rightarrow \dot{w} = P^{-1}\dot{z}$$
$$z = Pw$$
$$\dot{z} = Az$$
$$P\dot{w} = APw \Rightarrow \dot{w} = P^{-1}APw = \Lambda w,\ \dot{w}_i = \lambda_i w_i$$

<u>Suppose roots are real</u>

$$w_i(t) = w_i(0)e^{\lambda_i t}$$

$$\underset{n x 1}{w(t)} = \underset{n x n}{e^{\Lambda t}} \ \underset{n > 1}{w(0)}$$

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda n t} \end{bmatrix}$$

Careful: $e^{At} \neq \begin{bmatrix} e^{a_{11} t} & \ldots & e^{a_{1n} t} \\ \ddots & \ddots & \\ \ldots & \ldots & e^{a_{nnt}} \end{bmatrix}$

use

$$w(t) = e^{\Lambda t} w(0)$$

$$w = P^{-1}z, \text{ so } w(0) = P^{-1}z(0)$$

$$P^{-1}z(t) = \mathrm{e}^{\Lambda t}P^{-1}z(0)$$

$$z(t) = P e^{\Lambda t} c$$

where $w(0) \equiv c = P^{-1}z(0) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$

(3) $$z(t) = c_1 P_1 e^{\lambda_1 t} + c_2 P_2 e^{\lambda_2 t} + \ldots c_n P_n e^{\lambda_n t}$$

Conclude: A is stable iff $\lambda_i \leq 0$
A is asym. stable iff $\lambda_i < 0$
A is unstable if some $\lambda_i > 0$

Suppose some roots are imaginary: $\lambda_i = a \pm bi$

e.g.

$$\dot{z} = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} z$$

$$0 = \begin{vmatrix} 1-\lambda & -1 \\ 5 & -3-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 2 \Rightarrow$$

$$\lambda = \frac{-2 - \pm(4-8)^{\frac{1}{2}}}{2} = -1 \pm i \qquad i = \sqrt{-1}$$

Example for n dimensional case. Suppose $\lambda_1 = \lambda + \mu i$, $\lambda_2 = \lambda - \mu i$ $\lambda_3 \ldots \lambda_n$ are real.
Can show eigenvectors $P_1 = a + bi$, $P_2 = a - bi$, where a and b are vectors. (Eigenvectors associated with complex eigenvalues are complex.)

(4) $$z(t) = c_1\theta(t) + c_2\rho(t) + c_3 P_3 e^{\lambda_3 t} + \ldots + c_n P_n e^{\lambda n t}$$

$$\theta(t) = e^{\lambda t}\,(\mathrm{a} \cos \mu t - b \sin \mu t)$$
$$\rho(t) = e^{\lambda t}(\mathrm{a} \cos \mu + b \sin \mu t)$$

Point: stability depends on real part of complex eigenvalue.

Suppose $\nexists$ a set of linearly independent eigenvectors, i.e., P is singular.

e.g.

$$\dot{z} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} z$$

$$0 = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} \Rightarrow \lambda_1 = \lambda_2 = 2$$

To find P, solve $\begin{pmatrix} 1-2 & -1 \\ 1 & 3-2 \end{pmatrix} \begin{pmatrix} p_{11} \\ p_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{aligned} -p_{11} - p_{12} &= 0 \\ p_{11} + p_{12} &= 0 \end{aligned} \Rightarrow$$

$\exists$ a single eigenvector which is proportional to $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(see Boyce and Diprima page 294 for example of repeated eigenvalue with linearly independent eigenvectors.)

If eigenvalue is of multiplicity 2, and only 1 linearly independent eigenvector, P_1, the solution to ODE is of form

(5)
$$z_t = c_0 P_1 e^{\lambda t} + c_1 P_1 t e^{\lambda t} + \underset{\substack{\uparrow \\ \text{see B\& D for details on calculation}}}{c_2 \eta e^{\lambda t}}$$

Note that for all cases stability depends on sign of (real part of) λ_i

A system may converge for some I.C. in region of steady state. The set of initial conditions (a line or hyperplane) from which the system converges is called the stable manifold.

Suppose $\lambda_1, \lambda_2 ... \lambda_m < 0$, $\lambda_{m+1}, ... \lambda_n \geq 0$****

If I set $c_{m+1} ... c_n = 0$, (3) becomes

(3′)
$$z(t) = c_1 P_1 e^{\lambda_1 t} + ... c_m p_m e^{\lambda_m t} + 0 ... + 0$$

This converges, $\Rightarrow$

****In this case, equil. is a saddle point

$$z_0 = \left[P_1; P_2 \ldots P_m; P_{m+1} \ldots P_n\right] \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

recall c $\equiv P^{-1}z(0)$

This says that system converges if initial condition is a linear combination of eigenvectors associated with stable eigenvalues.

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A <u>saddle point</u> is an equilibrium which is approached from some region, not from others. (A saddle point has some + and some - roots.)

Some useful facts: trace $A = \Sigma\ \lambda_i$

$|A| = \Pi\ \lambda_i$

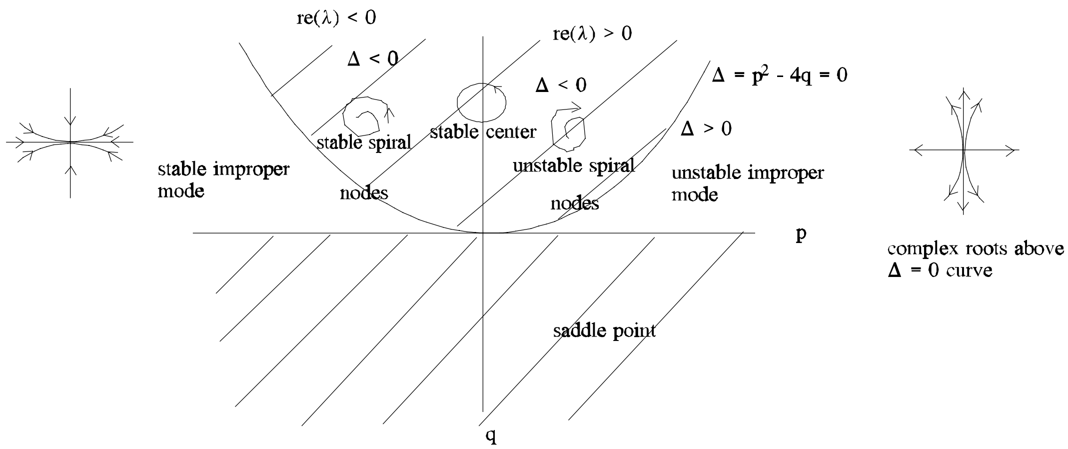
Pg 409 Boyce and Diprima

Two dimensional systems

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{aligned} p &\equiv a + d = \text{trace} \\ q &= ad - bc = |A| \end{aligned}$$

our rules $\Rightarrow \lambda_1 + \lambda_2 = p$

$\lambda_1 \lambda_2 = q$



$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - p\lambda + q\lambda \qquad \lambda = p \pm \frac{\sqrt{p^2 - 4q}}{2}$$

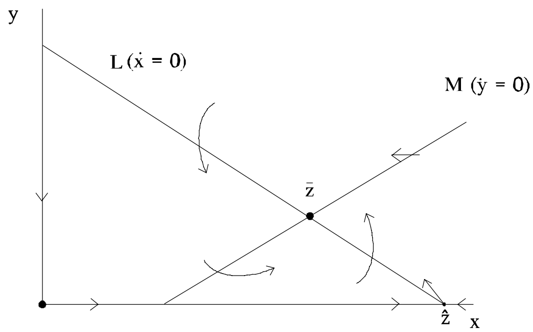
p = trace (A), q = determinate (A)
$\Delta \equiv p^2 - 4q$. If $\Delta < 0$, roots are complex

Summary: For 2 dimensional systems we can determine local stability by examing roots of linear system.

Return to predator-prey example, from page 7 of notes

prey $\dot{x} = (A - By - \lambda x)\, x \equiv f(x,y)$
predator $\dot{y} = (cx - D - \mu y)\, y \equiv g(x,y)$

L: $A - By - \lambda x = 0$
M: $Cx - D - \mu y = 0$



			$\bar{z}$	$\hat{z}$
f_x	=	$A\text{-}By\text{-}\ 2\lambda x$	$-\lambda\bar{x}$ *****	$A\text{-}2\lambda\hat{x}$
f_y		$-Bx$	$-B\bar{x}$	$-B\hat{x}$
g_x		Cy	$C\bar{y}$	0
g_y		$Cx\text{-}D\text{-}2\mu y$	$-\mu\bar{y}$	$C\hat{x}\text{-}D$

at $\bar{z}$

$$\dot{z} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \approx \begin{bmatrix} -\lambda\bar{x} & -B\bar{x} \\ C\bar{y} & -\mu\bar{y} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$p = -\lambda\bar{x} - \mu\bar{y} < 0, \qquad q = \lambda\mu\overline{xy} + Bc\overline{xy} > 0$$

$p < 0$, $q > 0$ $\Rightarrow$ stable (If the trace is negative, at least one root must be negative. Since the determinant is positive, both roots have the same sign. Therefore both roots must be negative.) Conclude that $\bar{z}$ is asymptotically stable.

***** (use $A - B\bar{y} - \lambda\bar{x} = 0$)

At $\hat{z}$ $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} \approx \begin{bmatrix} A-2\lambda\hat{x} & -B\hat{x} \\ 0 & c\hat{x}-D \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

(1,1) term = $-\lambda\hat{x} = -A < 0$ (use $A - \lambda\hat{x} = 0$)

(2,2) term = $\dfrac{CA}{\lambda} - D > 0$ (since $\hat{z}$ is below M)

$$q = (-)\cdot(+) < 0 \;\Rightarrow\; \text{saddle point}$$

Any point that starts on x axis converges to $\hat{z}$. The x axis is stable manifold.

An example where linearization does not tell us whether nonlinear system is stable is Voleterra-Lotke: $\lambda = \mu = 0$ (no upper bound on equil., e.g. if $y = 0$, prey grows without bound.)

Can show by linearizing system around steady state that ∃ purely imaginary eigenvalues. In this case, linear system is stable, but not asymptotically stable, so analysis of linear system does not tell us whether steady state of nonlinear system is a center or a spiral (focus). However, this system can be solved explicitly (See Clark).

Saddle point equilibria are important in optimal control problems. Our standard problem involves one "state variable", such as a stock of fish, where the initial condition is given. The solution of the control problem gives us an ODE for another variable, either the shadow value of the state, or equivalently, the optimal control (e.g. harvest). Thus we will analyze systems of two differential equations, consisting of the state variable and either the shadow value or the control variable. For autonomous problems, the state variable will converge to a steady state, and therefore, so must the other variable (the shadow value, or the control, depending on how we analyze the problem). We have to pick the initial value of the other variable (shadow value, or control) in order to drive the system to a steady state. If the steady state were unstable, there would be no way to get the system to converge. If the steady state were asymptotically stable, the system would converge no matter what the initial shadow value was. The only remaining possibility is for the steady state to be a saddle point. That means that the system will converge only if we pick the initial control correctly.

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Two examples of analysis of two-dimensional dynamic systems:

1) Brander, James and Scott Taylor "The Simple Economics of Easter Island: a Ricardo-Malthus Model of Renewable Resource Use". AER 1998, pp 119 - 138.

They use a simple general equilibrium, renewable resource model to describe the rise and fall of an isolated civilization, like the one on Easter Island. The economy produces two goods and has Ricardian technology. "The Environment" (a stock variable) affects the marginal productivity of labor in one sector. The other state variable is the stock of human population. An increase in the environment encourages growth of human population (it becomes easier to get food). An increase in human population degrades the environment, since extraction increases. Authors use the predator-prey model described in these notes, with the environment as "the prey" and mankind as "the predator". Analysis proceeds by linearizing dynamics around the interior steady state and checking whether the roots are real or imaginary. They show that a stable interior equilibrium can be either an "improper node" (real roots) with monotonic adjustment or a spiral node (imaginary roots), with trajectories spiralling into the steady state. Spirals occur if the intrinsic growth rate of the environment is sufficiently small. Using "reasonable" parameter values, they show that it is possible to get a trajectory in which human population increases for a time, then crashes, eventually reaching a low level.

2) Krugman, P. (1991) "History versus Expecation" QJE 651 - 67. (see also the 1993 QJE article by Benabu and Fukao which corrects a technical error in Krugman's analysis.)

Two sector economy, one sector has increasing returns to scale. Hold output prices fixed (e.g., small open economy), consider allocation of labor between sectors. Let L be the amount of labor in IRTS sector, 1 = total amount of labor, so 1 - L is amount of labor in CRTS sector. The VMP = wage in IRTS sector is $a + bL$, $b > 0 \Rightarrow$ ICTS. The VMP of labor in CRTS sector $\equiv c$. a, b, c are constants.

Suppose that cost of moving from between sectors is $\gamma[(\dot{L})^2]/2$ where $\gamma > 0$. The marginal cost of moving is therefore the absolute value of $\gamma\dot{L}$. An individual that migrates at time t pays the price equal to marginal cost. (You can think of "migration services" being competitively supplied.) In this model there are convex adjustment costs: the marginal cost of moving increases with the speed at which movement takes place.

The wage differential at a point in time is $m(L) = a - c + bL$. Assume $c > a$, $a+b > c$. If all labor is in IRTS sector, wage there is higher. If no labor is in IRTS sector, wage there is lower.

Suppose, for example, that an individual migrates from the CRTS sector to the IRTS sector at time t. We have seen that this individual pays the price (the abolute value of) $\gamma\dot{L}$. How much does the individual gain? Define T as the time at which all migration ceases. (T may be infinite, but in this problem it happens to be finite.) An instant after T, an individual can migrate without paying any costs, since by definition no one else is migrating at that time, so $\dot{L} = 0 = \gamma\dot{L}$ (= the price of migration). By migrating at t rather than T, the individual gains q, defined as

$$q = \int_0^T e^{-rt} m(t) dt \quad \Rightarrow \quad \dot{q} = rq - m \ .$$

The variable q obeys $\dot{q} = rq - m$. Equation of this sort appear in many contexts. Interpret this funadamental "no arbitrage condition": Dividend (the wage differential) plus capital gain ($\dot{q}$) = opportunity cost of holding the stock (rq).

If an individual worker takes all prices (the cost of moving and the wage differential at every point in the future) as given and has rational expectations, equilibrium requires

$$\gamma \dot{L} = q$$

The equilibrium condition simply says that an individual is willing to pay "what it is worth" to get into a different sector.

We can write the equilibrium conditions for this model as

$$\begin{pmatrix} \dot{q} \\ \dot{L} \end{pmatrix} = \begin{bmatrix} r & -b \\ 1/\gamma & 0 \end{bmatrix} \begin{pmatrix} q \\ L \end{pmatrix}$$

The trace = r > 0 and the determinant = $b/\gamma > 0$, and the discriminat is $\Delta = r^2 - 4b/\gamma$. Since both the trace and the determinant are positive, we know that the steady state, L^*, is unstable. If $\Delta < 0$ (i.e. if r or γ are small, or if b is large) the steady state is an unstable node; if $\Delta > 0$ the steady state is an unstable node.

Figures (13) and (14) show the two possible phase portraits of these two differential equations. In both cases, there are three steady states, the boundaries L = 0 and L = 1, and the interior equilibrium denoted L*. We saw that the interior steady state is unstable. Instability occurs because of IRTS; discuss economic intuition.

L* could be an unstable spiral, as shown in figure 13 or an unstable node as shown in figure 14. The dashed trajectory (figure 13) leads to L = 0, the solid trajectory leads to L = 1. For initial conditions ε (L_1, L_2) economy could be on either trajectory, so "expectations matter" (i.e., expectations determine trajectory of economy). For initial conditions less than L_1 or greater than L_2 there is only one equilibrium trajectory, so here "history matters.

If L* is an unstable node (figure 14), only history matters. For initial conditions $L < L^*$,

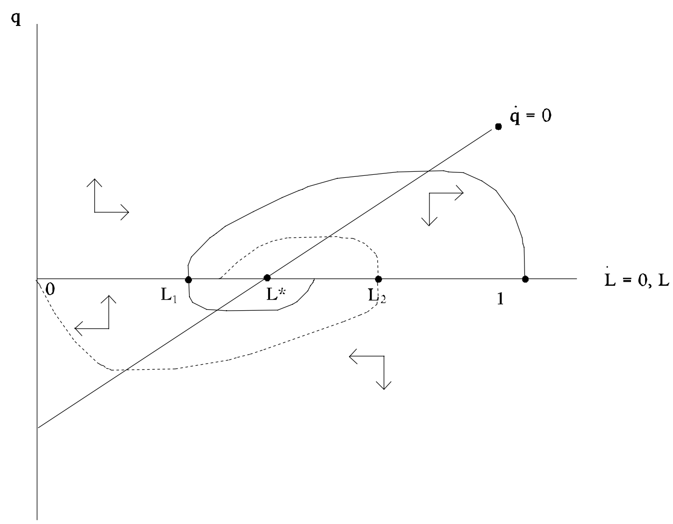

Figure 13 Phase space of L,q

economy moves to low steady state, for initial conditions $L > L^*$, economy moves to high steady state.

Property of unstable equilibrium depends on parameters (which determine the value of Δ). Unstable node iff r is large relative to b and γ.