In class midterm Exam - Answer key

Problem 1 (12 points). Convergence:

Given two sequences $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ that converge to points $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ in $\mathbb{R}^n$ prove that the sequence $\{\mathbf{x}_n + \mathbf{y}_n\}$ converges to the point $(\bar{\mathbf{x}} + \bar{\mathbf{y}})$. Hint: for this problem it's easiest by far to work with the metric $d(\mathbf{v}, \mathbf{w}) = |\mathbf{v} - \mathbf{w}|$, which is equivalent to the Pythagorian metric.

Ans: Fix $\epsilon > 0$. Since $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ converge to points $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ in $\mathbb{R}^n$, there exist $N^{\mathbf{x}}, N^{\mathbf{y}}$ such that for $n > N^{\mathbf{x}}$, $d(\mathbf{x}_n, \bar{\mathbf{x}}) < \epsilon/2$ while for $n > N^{\mathbf{y}}$, $d(\mathbf{y}_n, \bar{\mathbf{y}}) < \epsilon/2$. Therefore, by the triangle inequality

$$\begin{aligned} d(\mathbf{x}_n + \mathbf{y}_n, \bar{\mathbf{x}} + \bar{\mathbf{y}}) &= |(\bar{\mathbf{x}} + \bar{\mathbf{y}}) - (\mathbf{x}_n + \mathbf{y}_n)| \\ &= |\bar{\mathbf{x}} - \mathbf{x}_n + \bar{\mathbf{y}} - \mathbf{y}_n| \leq |\bar{\mathbf{x}} - \mathbf{x}_n| + |\bar{\mathbf{y}} - \mathbf{y}_n| < \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

Problem 2 (12 points). Extreme values:

Let S be some closed subset of $\mathbb{R}^2$ with the property that for all $s \in S$, $||s|| \leq 19$.

a) Prove that there is a point $s^* \in S$, which is closest to the origin in the Euclidean norm, n (in other words, $n^E(s^*) \leq n^E(s)$ for all $s \in S$). You may assume that for every open set $O \in \mathbb{R}^2$, the set $\{s \in S : n^E(s) \in O\}$ is open.

Ans: S is closed and bounded and hence compact. Since for every open set $O \in \mathbb{R}^2$, the set $\{s \in S : n^E(s) \in O\}$ is open, we know that n^E is continuous. Therefore, by the Weierstrass theorem, n^E attains a minimum, i.e., there exist $s^* \in S$ s.t. $n^E(s^*) \leq n^E(s)$ for all $s \in S$.

b) Show with a counter example that this statement in part a) would not necessarily be true if $\mathbb{R}^2$ were endowed with some metric other than the Pythagorian metric. (The norm n is the same as in part a); the only thing that changes, moving from a) to b) is the metric on $\mathbb{R}^2$.)

Ans: Let d be the discrete metric on $\mathbb{R}^2$. Let S be the set $\{s \in \mathbb{R}^2 : 0 < ||s|| \leq 19\}$. Since every subset of $\mathbb{R}^2$ is closed under the discrete metric, S is closed. For any $s \in S$, $0.5s \in S$, and $||0.5s|| = 0.5||s||$. Hence there is no point in S satisfying the specified property.

Problem 3 (17 points). Sequences, etc:

Given an arbitrary metric d and set $A \subset \mathbb{R}$

a) Prove that a point $x \notin A$ is a boundary point of A iff it is an accumulation point of A.

Ans: Suppose that x is a boundary point of A but $x \notin A$. Then for all $n \in \mathbb{N}$ there exists $y_n \in A \cap B(x, 1/n)$ and $z_n \in (R \backslash A) \cap B(x, 1/n)$. Thus $y_n \to x$, but since $x \notin A$, $y_n \neq x$ for all n. Hence x is an accumulation point of A.
Now assume that x is an accumulation of A but $x \notin A$. Then for all $n \in \mathbb{N}$, there exists $y \in B(x, 1/n) \cap A$. Moreover, for all n, $x \in B(x, 1/n) \cap (R \backslash A)$. Hence x is a boundary point of A.

b) Prove that a sequence in A can have at most one limit point.

Ans: Suppose that (x^n) is a sequence in A that has a limit point $\bar{x} \in A$. Consider $x \neq y \in A$. There exists $\epsilon > 0$ such that $d(x,y) > 2\epsilon$. Since (x^n) converges to $\bar{x}$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $d(x,x_n) < \epsilon$. By the triangle inequality $d(x,y) \leq d(x,x_n) + d(x_n,y)$, or $d(x_n,y) \geq d(x,y) - d(x,x_n) > 2\epsilon - \epsilon = \epsilon$. We have established then, that for all $n > N$ $d(x_n,y) > \epsilon$ so that (x_n) cannot converge to y. Since y was chosen arbitrarily, this proves that (x_n) cannot have more than one limit point.

Problem 4 (17 points). Continuity:

Let f be a continuous function, $f : [a,b] \to \mathbb{R}$, where $[a,b]$ is a closed interval in $\mathbb{R}$. Prove that $f([a,b])$ is a closed interval.

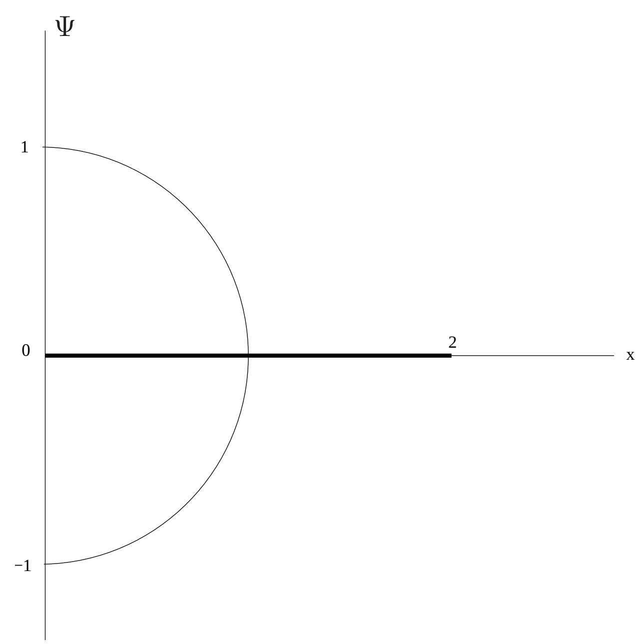
Ans: Let $F = f([a,b])$ and let α be an accumulation point of F. We need to show that $\alpha \in F$. By assumption, there exists a sequence (α_n) in F such that $\lim_n \alpha_n \to \alpha$. For each $n \in \mathbb{N}$, since $\alpha_n \in F$, there exists $x_n \in [a,b]$ such that $f(x_n) = \alpha_n$. Since $[a,b]$ is compact, there is a subsequence (x_{n_k}) of (x_n) that converges to $x \in [a,b]$. Since f is continuous, the sequence $(f(x_{n_k}))$ converges to $f(x)$. But $f(x_{n_k}) = \alpha_{n_k}$ and $\lim_k \alpha_{n_k} = \alpha$. Since a sequence can only have one limit, $\alpha = f(x)$. Hence $\alpha \in F$.

Problem 5 (17 points). Hemi-Continuity:

Let the correspondence $\Psi : [0,2] \twoheadrightarrow \mathbb{R}$ be defined as $\Psi(x) = \{y | x^2 + y^2 = 1\}$

a) Sketch the graph of Ψ.

Ans:



b) Is Ψ upper hemicontinuous? If yes explain briefly why; a full proof is not required though would be appreciated if you have time. If no, prove by example that it is not UHC.

Ans: It's upper hemi-continuous. For any open set $O \in \mathbb{R}$,
$\bar{\Psi}^{-1}(O) = \{x \in [0,2] : x = \pm\sqrt{1-y^2}\}$. Since O is open, this set is open.

c) Is Ψ lower hemicontinuous? If yes explain briefly why; a full proof is not required though would be appreciated if you have time. If no, prove by example that it is not LHC.

Ans: It's not lower hemi-continuous. Let $O = (-1,1)$. $\underline{\Psi}^{-1}(O) = (0,1]$ which is not open.

Problem 6 (25 points). Vector Spaces:

Let P_3 be the set of all third-order polynomials defined on $\mathbb{R}$ (in other words, all polynomials of the form $p^0 = a + bx + cx^2 + dx^3$, where $x, a, b, c, d \in \mathbb{R}$.)

a) Show that P_3 is a vector space.

Ans: Let $p^1 = a^1 + b^1 x + c^1 x^2 + d^1 x^3$ and $p^2 = a^2 + b^2 x + c^2 x^2 + d^2 x^3$. For $\alpha, \beta \in \mathbb{R}$, let

$$p = \alpha p^1 + \beta p^2 = (\alpha a^1 + \beta a^2) + (\alpha b^1 + \beta b^2)x + (\alpha c^1 + \beta c^2)x^2 + (\alpha d^1 + \beta d^2)x^3$$

Since $(\alpha a^1 + \beta a^2)$, $(\alpha b^1 + \beta b^2)$, $(\alpha c^1 + \beta c^2)$ and $(\alpha d^1 + \beta d^2)$ are all in $\mathbb{R}$, $p \in P_3$. This verifies that P is a vector space.

b) Write down a basis for P_3 . What is the dimension of P_3? Write down the vector of weights such that using these weights the linear combination of the basis set you have chosen is p^0

Ans: A basis for P_3 is $\{1, x, x^2, x^3\}$. This has four elements, so that the dimension of P_3 is 4. The vector of weights is $\begin{bmatrix} a, & b, & c, & d \end{bmatrix}$, i.e.,

$$p0 = \begin{bmatrix} a, & b, & c, & d \end{bmatrix} \cdot \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix}$$

c) Let T be a linear transformation defined by $T[f(\cdot)] = f'(x) - f''(x)$. A basis set for $T(P_3)$ is $\{1, x, x^2\}$.

i) Write down the vector of weights in $\mathbb{R}^3$ such that the linear combination of $\{1, x, x^2\}$ with these weights is $T(p^0)$.

Ans:

$$T(p^0) = b + 2cx + 3dx^2 - (2c + 6dx) = (b - 2c) + (2c - 6d)x + 3dx^2.$$

so the vector of weights is $\begin{bmatrix} b - 2c, & 2c - 6d, & 3d \end{bmatrix}$.

ii) Write down the matrix M that expresses T in terms of the basis you wrote down in b) and the basis set $\{1, x, x^2\}$. In other words, if v^i is the i'th element of the basis set you wrote down in b), then $T(v^i)$ should equal the linear combination of $\{1, x, x^2\}$ with weights equal to the i'th column of M.

Ans: The matrix is

$$M = \begin{bmatrix} 0 & 1 & -2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

iii) Verify that Mp^0 equals the answer you got in part i) of c);

Ans:

$$\begin{bmatrix} 0 & 1 & -2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} b-2c \\ 2c-6d \\ 3d \end{bmatrix}$$

Problem 7 (25 points). Matrices:

Let (M^n) be a sequence of 2×2 symmetric matrices. Assume that

(i) there exists $\alpha \in (-1, 0)$ such that for all n, $\det(M^n) = \alpha$.

(ii) there exists unit length vectors $\mathbf{v}^1, \mathbf{v}^2 \in \mathbb{R}^2$ s.t. for all n, $\mathbf{v}^1, \mathbf{v}^2$ are eigen-vectors for M^n.

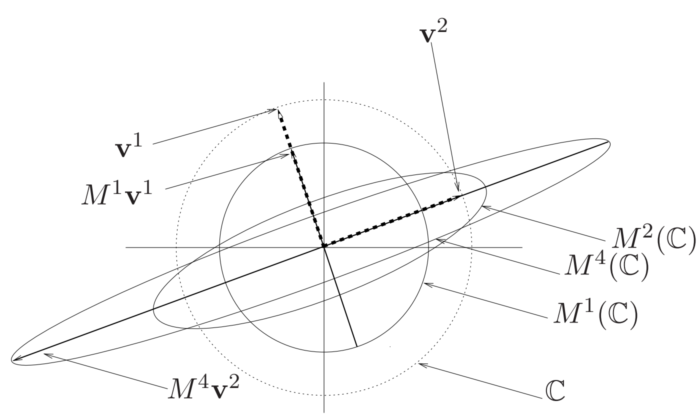
(iii) For each n, let $\lambda_1^n > 0$ be the eigenvalue corresponding to $\mathbf{v}^1$ for the matrix M^n. Assume that (λ_1^n) is a strictly decreasing sequence with $\lim_n \lambda_1^n = 0$.

(iv) For each n, denote by λ_2^n the eigenvalue corresponding to $\mathbf{v}^2$ for the matrix M^n. Assume that $|\lambda_2^1| = |\lambda_1^1|$, i.e., the eigenvectors have the same absolute value for M^1.

Questions:

a) Sketch the image of the unit circle $\mathbb{C}$ under M^n for three values of n including $n = 1$, say, for example $n = 1, 2, 4$. Your graph should reflect what you know about α. Include a sketch of the unit circle on your figure, for reference. Also, indicate the image of the eigenvectors on your graph. Label everything on your graph clearly so there's no ambiguity about what is what.

Ans:



b) What can you say about the definiteness of each M^n?

Ans: Since $\alpha < 0$, there must be one positive and one negative eigenvalue. So each matrix is indefinite.

c) Does (λ_2^n) contain a convergent subsequence? Justify your answer.

Ans: We know that for each n, $\det(M^n) = \prod_{i=1}^2 \lambda_i^n = \alpha$. Hence $\lambda_2^n = \alpha/\lambda_1^n$. Since $(\lambda_1^n) \to 0$, (λ_2^n) increases without bound. Hence it cannot contain a convergent subsequence.

d) Let $P_+^n = \{\mathbf{x} \in \mathbb{C} : \mathbf{x}' M^n \mathbf{x} > 0\}$ and $P_-^n = \{\mathbf{x} \in \mathbb{C} : \mathbf{x}' M^n \mathbf{x} < 0\}$. What is:
(i) $\cup_{n=1}^\infty P_+^n$? (ii) $\cap_{n=1}^\infty P_+^n$? (iii) $\cup_{n=1}^\infty P_-^n$? (iv) $\cap_{n=1}^\infty P_-^n$?
You may prefer to draw a picture to illustrate some or all of these.

Ans: Answers not in order:

(ii) Since (λ_1^n) is positive and shrinking to zero, the sets P_+^n decrease with n. The *only* vector that belongs to P_+^n, for all n is $\mathbf{v}^1$. Thus $\cap_{n=1}^\infty P_+^n = \{\mathbf{v}^1, -\mathbf{v}^1\}$

(iii) Eventually, all vectors get pulled toward $\mathbf{v}^2$, the eigenvector with the negative eigenvalue that increases without bound. Hence Thus $\cup_{n=1}^\infty P_-^n = \mathbb{C} \backslash \{\mathbf{v}^1, -\mathbf{v}^1\}$

(i) The matrix M^1 has two equal eigenvalues. So the eigenvectors $\mathbf{v}^1$ and $\mathbf{v}^2$ "pull" the other vectors with "equal force." Thus, P_+^1 contains all vectors in $\mathbb{C}$ that are closer to $\pm\mathbf{v}^1$ than to $\pm\mathbf{v}^2$, i.e., P_+^1 contains almost half the circle. Vectors equi-distantfrom $\mathbf{v}^1$ and $\mathbf{v}^2$ are *not* in P_+^1. Since (λ_1^n) is a decreasing sequence, the P_+^1's are nested and $\cup_{n=1}^\infty P_+^n = P_+^1$.

(iv) The matrix M^1 has two equal eigenvalues. So the eigenvectors $\mathbf{v}^1$ and $\mathbf{v}^2$ "pull" the other vectors with "equal force." Thus, P_-^1 contains all vectors in $\mathbb{C}$ that are closer to $\pm\mathbf{v}^2$ than to $\pm\mathbf{v}^1$, i.e., P_-^1 contains almost half the circle. Vectors equi-distantfrom $\mathbf{v}^1$ and $\mathbf{v}^2$ are *not* in P_-^1. Since (λ_2^n) is an increasing sequence, P_-^1 is the smallest of the P_-^n's, so that $\cap_{n=1}^\infty P_-^n = P_-^1$.