Midterm Exam - Answer key

Problem 1: Continuity. (14 points)

In the first two parts of this problem, determine whether or not the specified functions are continuous. If they are, prove it. If they are not, identify the subset X in the domain of the function such that for every $x \in X$, f is continuous at x.

a)

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Ans: A function $f : X \to \mathbb{R}^k$ is continuous at a point $x_0 \in X$ if $\forall \epsilon > 0$, there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$. The function f is continuous if it is continuous at x, for every $x \in X$.

At $x_0 = 0$: Fix $\epsilon > 0$ and set $\delta = \epsilon$. When $|x - 0| < \delta$, then

$$|f(x) - f(0)| = \begin{cases} |x - 0| < \epsilon & \text{if } x \text{ rational} \\ |0 - 0| < \epsilon & \text{if } x \text{ irrational} \end{cases}$$

Therefore the function is continuous at $x = 0$.

At $x_0 \neq 0$, where x_0 is rational: in this case, $f(x_0) = x_0$ so that

$$|f(x) - f(x_0)| = \begin{cases} |x - x_0| & \text{if } x \text{ rational} \\ x_0 & \text{if } x \text{ irrational} \end{cases}$$

Set $\epsilon = 0.5x_0$ and choose $\delta > 0$ arbitrarily. Regardless of the size of δ, we can choose an irrational x such that $|x - x_0| < \delta$; we then have $|f(x) - f(x_0)| = x_0 > \epsilon$. Hence f is not continuous at x_0.

At $x_0 \neq 0$, where x_0 is irrational: In this case, $f(x_0) = 0$ and

$$|f(x) - f(x_0)| = \begin{cases} |x| & \text{if } x \text{ rational} \\ 0 & \text{if } x \text{ irrational} \end{cases}$$

Once again, set $\epsilon = 0.5x_0$, and pick a rational $x \in (0.5x_0, x_0)$, $|f(x) - f(x_0)| = x > 0.5x_0 = \epsilon$. Hence f is not continuous at x_0.

Conclude that f is continuous only at $x_0 = 0$

b)

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/q & \text{if } x = p/q \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Ans: At $x_0 = 0$: In this case $f(x_0) = 1$. Pick $\epsilon = 0.5$ and choose $\delta > 0$ arbitrarily. There exists an irrational x such that $|x| < \delta$, and for this x $|f(x) - f(x_0)| = 1 > \epsilon$. Hence f is not continuous at x_0.

At $x_0 = p/q$ rational: In this case, $f(x_0) = 1/q$. Pick $\epsilon = 1/2q$, choose $\delta > 0$ arbitrarily and pick an irrational $xinB(x_0, \delta)$ Since $|f(x) - f(x_0)| = 1/q > \epsilon$, f is not continuous at x_0.

At x_0 irrational: In this case, $f(x_0) = 0$. Fix $\epsilon > 0$ and define $q(\epsilon) = \min\{q \in \mathbb{N} : 1/q < \epsilon\}$. Next, define $p(\epsilon) = \min\{p \in \mathbb{N} : p/q(\epsilon) > x_0\}$. Now define $R = \{x = p/q : p \leq p(\epsilon); q \leq q(\epsilon)\}$. Since R is a finite set which does not contain x_0 (since, by assumption, x_0 is irrational), we can pick $\delta > 0$ sufficiently small such that $B(x_0, \delta) \cap R$ is empty. Finally, note that if $x = p/q \in (B(x_0, \delta) \cup R^C)$, then $q > q(\epsilon)$. (This it true because if $q \leq q(\epsilon)$ and $p/q \notin R$ then necessarily $p > p(\epsilon)$, and hence $p/q \geq p/q(\epsilon) > p(\epsilon)/q(\epsilon) > x_0 + \delta$). It now follows that $x \in B(x_0, \delta)$ implies $|f(x) - f(x_0)| = |f(x)| < \epsilon$: if x is irrational, then $f(x) = 0 < \epsilon$; if x is rational, then, as we have shown, $x = p/q \in B(x_0, \delta)$ implies $q > q(\epsilon)$ and hence $f(x) < 1/q(\epsilon) < \epsilon$. We have proved, therefore, that $x \in B(x_0, \delta)$ implies $|f(x) - f(x_0| < \epsilon$, verifying that f is continuous at x_0.

Conclude that f is continuous at every irrational number, and discontinuous at every rational number.

c) To do this part, you need the formal definition of the derivative of a function. Let $g : \mathbb{R} \to \mathbb{R}$ and pick $x \in \mathbb{R}$. The derivative of g at.x is defined as:
$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$, $(h \in \mathbb{R})$. Now define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} x^n \sin(x^{-m}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

where n and m are natural numbers. Identify conditions on n and m that are necessary and sufficient for the following properties:

i) g is differentiable at 0.

Ans: From the definition of a derivative, observe that
$g'(0) = \lim_{x \to 0} \frac{x^n \sin(x^{-m}) - 0}{x - 0} = \lim_{x \to 0} x^{n-1} \sin(x^{-m})$.
Since $m \geq 1$, x^{-m} increases without bound as $x \to 0$. If $n = 1$, so that $x^{n-1} \equiv 0$, then $\lim_{x \to 0} x^{n-1} \sin(x^{-m}) = \lim_{x \to 0} \sin(x^{-m})$ does not exist. If $n > 1$, $x^{n-1} \to 0$ as $x \to 0$. Since $\sin(x^{-m}) \in [-1, 1]$, for all x, $\lim_{x \to 0} x^{n-1} \sin(x^{-m}) = 0$. Therefore a necessary and sufficient condition for g to be differentiable at 0 is that $n > 1$.

ii) $g'(\cdot)$ is continuous at 0.

Ans: Clearly, $g'(\cdot)$ cannot be continuous at zero unless g is differentiable at zero. Hence a necessary condition for $g'(\cdot)$ to be continuous at zero is that $n > 1$. Assume now that $n > 1$ and note that for $x \neq 0$, $g'(x) = nx^{n-1} sin(1/x^m) - mx^{n-m-1} cos(1/x^m)$. Since $g'(0) = 0$, $g'(\cdot)$ will be continuous at zero iff $\lim_{x \to 0} g'(x) = 0$. This property will be satisfied iff $x^{n-m-1} \to 0$ as $x \to 0$, a condition which will be satisfied iff $(n - m - 1)$ is positive i.e., iff $n > m + 1$.

Problem 2: Metrics. (14 points)

a) Let x and y be two differentiable real valued functions. Is $d(x, y) = \int_{-\infty}^{\infty} |x(t) - y(t)|\, dt$ a metric? if it is, prove it. If not, explain why and provide a counter-example.

Ans: It is not a metric. To see this, let x(t)=t+1 and y(t)=t. Both functions are real valued and differentiable. But $d(x, y) = |x(t) - y(t)| = \int_{-\infty}^{\infty} |x(t) - y(t)|\, dt = \infty$. Since a metric is required to be a real-valued function, d is not a metric.

b) Let $f : \mathbb{R} \to \mathbb{R}$ and let $f^n(x)$ denote the n-th derivative of the function f with respect to x. For any point $x \in \mathbb{R}$, let $y_x^n = f^n(x)$. Find conditions on f so that $\{y_x^n\}$ converges in the Pythagorean metric for every $x \in \mathbb{R}$. Note that there are lots of sufficient conditions, many of them silly (e.g., a sufficient condition is that f is the zero function). To discourage silly answers like this, we're going to give you more marks, the larger is the set of functions that satisfies your sufficient condition.

Ans: Let $f^n(x) = g(x)$. Assume that the distance between $g(x)$ and $g'(x)$ is ϵ. For simplicity, assume $g'(x) - g(x) = \epsilon$. Solving this first order differential equation we obtain $g(x) = e^{x+b} - \epsilon$. Since the definition of convergence requires the difference to be less than any $\epsilon > 0$, then make $g(x) = e^{x+b}$. The derivative of this function is the function itself. Integrating, define $f(x) = e^{x+b} + c$. In general, g(x) can be the nth derivative of a function. Integrating we get a general function of the form $f(x) = (\sum_1^n a_n x^n)(e^{x+b} + c) + d$ where c and d are constants. The sequence $\{y_n\}$ converges to e^{x_0+b} for any x_0, b and c in $\mathbb{R}$. Another function can be a polynomial of degree n. Note that the n+1 th derivative is zero. Then the sequence converges to zero.

Problem 3: <u>More on metrics</u>. (14 points)

a) Let for $i = 1, \ldots n$, let d^i be a metric on $\mathbb{R}$. and define $\mathbf{d} = (d^1, \ldots, d^i \ldots, d^n)$. Now for some function f, define $d^f(x, y) = f(\mathbf{d}(x, y))$. Is $d^f(x, y)$ a metric? If so, prove it. If not, provide sufficient conditions on f for it to be a metric. Once again, the weaker the set of conditions you provide (i.e., the more f's that satisfy the conditions), the more points you get.

Ans: Note that the question only asks for sufficient conditions. Following the definition of a metric, Let d^* be a nonnegative function defined on $\mathbb{R}^n$, $d^* : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ such that for every x, y and z in $\mathbb{R}^n$ we have:
(i)$d^*(x, y) \geq 0$
(ii)$d^*(x, y) = 0$ iff $x = y$
(iii) $d^*(x, y) = d^*(y, x)$
(iv) $d^*(x, y) \leq d^*(x, z) + d^*(z, y)$
Following the definition of d^*:
(i)$d^*(x, y) = f(\mathbf{d}(x, y)) \geq 0$. Therefore the definition requires f to be such that $f : \mathbb{R}_+^n \to \mathbb{R}_+$.
(ii)If $x = y$ then $d^*(x, y) = f(\mathbf{d}(x, y)) = f(\mathbf{0})$. Therefore $f(\mathbf{0}) = 0$
(iii) $d^*(x, y) = f(\mathbf{d}(x, y)) = f(\mathbf{d}(y, x)) = d^*(y, x)$. Therefore no further requirements on f.
(iv) $d^*(x, y) = f(\mathbf{d}(x, y))$
Assume f is a strictly increasing function then
$f(\mathbf{d}(x, y)) \leq f(\mathbf{d}(x, z) + \mathbf{d}(z, y))$
Assume f a linear transformation, then

$$f(\mathbf{d}(x,z)+\mathbf{d}(z,y)) = f(\mathbf{d}(x,z)) + f(\mathbf{d}(z,y)) = d^*(x,z) + d^*(z,y)$$

b) Determine if either, neither or both $min(d_1, d_2)$ or $max(d_1, d_2)$ are metrics. If they are prove it. If not, provide a counter example.

Ans: $\max(d_1, d_2)$ is a metric but $\min(d_1, d_2)$ is not. To prove that $\max(d_1, d_2)$ is a metric, we will just check the triangle inequality. The other conditions are obviously satisfied. For $i = 1, 2$, we have, for all $x, y, z \in \mathbb{R}$.

$$\begin{aligned} d_i(x,y) &\leq d_i(x,z) + d_i(z,y) \\ &\leq \max(d_1(x,z), d_2(x,z)) + \max(d_1(z,y), d_2(z,y)) \end{aligned}$$

Hence

$$\max(d_1(x,y), d_2(x,y)) \leq \max(d_1(x,z), d_2(x,z)) + \max(d_1(z,y), d_2(z,y))$$

verifying that the triangle inequality is satisfied.
We now construct a counter example of two metrics on $\mathbb{R}^2$ and show that for these metrics $\min(d_1, d_2)$ fails the triangle inequality, hence $\min(\cdot,\cdot)$ is not a metric. For some $\epsilon \in (0, 0.25)$, let $d_1(x,y) = \sqrt{\epsilon(x_1-y_1)^2 + (x_2-y_2)^2}$ and $d_2(x,y) = \sqrt{(x_1-y_1)^2 + \epsilon(x_2-y_2)^2}$. (These are two examples of a metric called the *Mahalanobis metric*, which is a generalization of the Pythagorian metric: for the Pythagorian metric, the level sets of the metric are circles; for the Mahalanobis metric, they are arbitrary ellipses. (For pictures, see http://www.comp.lancs.ac.uk/ kristof/research/notes/basicstats/index.html). Now let $x = (0,0)$, $y = (1,1)$ and $z = (1,0)$, Note that $d_1(x,y) = d_2(x,y) = \sqrt{1+\epsilon}$. On the other hand, $d_1(x,z) = \sqrt{\epsilon} = d_2(z,y)$, while $d_2(x,z) = 1 = d_2(z,y)$. Hence $\min(d_1(x,z), d_2(x,z)) = \min(d_1(z,y), d_2(z,y)) = \sqrt{\epsilon}$. Since $\epsilon < 0.25$, it follows that

$$\min(d_1(x,y), d_2(x,y)) = \sqrt{1+\epsilon} > 2\sqrt{\epsilon} = \min(d_1(x,z), d_2(x,z)) + \min(d_1(z,y), d_2(z,y)).$$

Thus the triangle inequality fails for $\min(\cdot,\cdot)$.

c) Let

$$d(x,y) = \begin{cases} d^1(x,y) & \text{if } x > y \\ d^2(x,y) & \text{if } x \leq y \end{cases}$$

where $d^1(x,y)$ and $d^2(x,y)$ are metrics. Demonstrate that $d(x,y)$ is not a metric. A function very like d is, however, a metric. This function, call it ρ is defined almost identically to d, except that a small number of the symbols that define d (somewhere between 1 and 4 of them) have to be changed. Identify which symbols need to be changed, and verify that the newly defined function is indeed a metric.

Ans: Note that

$$d(x,y) = \begin{cases} d^1(x,y) & \text{if } x > y \\ d^2(x,y) & \text{if } x \leq y \end{cases}$$

and

$$d(y,x) = \begin{cases} d^1(x,y) & \text{if } y > x \\ d^2(x,y) & \text{if } y \le x \end{cases}$$

Then $d(x,y) \neq d(y,x)$. Therefore $d(x,y)$ is not a metric. If we change the inequality signs for $=$ and $\neq$. Then:

$$ld(y,x) = \begin{cases} d^1(x,y) & \text{if } y \neq x \\ d^2(x,y) & \text{if } y = x \end{cases}$$

Since $d^2(x,y) = 0$ when $x = y$ then $d(x,y) = d^1(x,y)$. Then $d(x,y)$ is a metric. Alternatively, If you choose:

$$d(y,x) = \begin{cases} d^1(x,y) & \text{if } y = x \\ d^2(x,y) & \text{if } y \neq x \end{cases}$$

Then $d(x,y) = d^2(x,y)$. Then $d(x,y)$ is a metric.

Problem 4: Basic Analysis. (14 points)

Let $S = \{(x,y,z)\,|x = sin\phi\cos\theta, y = sin\phi\sin\theta, z = cos\phi, \phi > 0, \theta > 0\}$
Hint: this problem is much easier if you use "polar coordinates." Check wikipedia or any other source for a definition of polar coordinates.

Ans: This is a sphere in $\mathbb{R}^3$ with radius 1 defined on spherical coordinates. The surface of a sphere is a curved finite space.

a) Let $f(\mathbf{x},\mathbf{y}) = \varphi(\mathbf{x},\mathbf{y})$ where $\mathbf{x}$ and $\mathbf{y}$ belong to S and $\varphi(\mathbf{x},\mathbf{y})$ is the minimum non negative angle between the two vectors measured in radians. 2Π radians equal 360 degrees. Define an ϵ ball on S in this metric.

Ans: $B_\epsilon(\mathbf{x}) = \{\mathbf{y}\,|\mathbf{y} \in S, \varphi(\mathbf{x},\mathbf{y}) < \epsilon\}$. The epsilon ball is given by all the points such that the angle between the vector going from the origin to x and the vector that goes from the origin to those points have an angle less than ϵ radians. This looks like a circle on the surface of the sphere.

b) Let $\{x_n\} = \{\sin(\pi/2^n + \pi n/2)\cos(\pi n), \sin(\pi/2^n + \pi n/2)\sin(\pi n), \cos(\pi/2^n + \pi n/2)\}$
Use the pythagorean metric. Show that this sequence is not a Cauchy sequence.

Ans: It is neither Cauchy or convergent. Note that $\cos(\Pi n) = (-1)^n$ and $\sin(\Pi n) = 0$ where n is a natural number. Therefore
$\{x_n\} = \{(-1)^n \sin(\pi/2^n + \pi n/2), 0, \cos(\pi/2^n + \pi n/2)\}$
To see that the sequence is not cauchy note that as n increases, $\pi/2^n$ approaches zero. Since the values both functions at a any given angle θ are different from the values at $\theta + \Pi/2$, then the components are different. Therefore, the angle of the vectors $\{x_n\}$ and $\{x_{n+1}\}$ is always positive. The function is not Cauchy.
To see that the function is not convergent, note that as n increases, $\pi/2^n$ approaches zero and

the functions $\sin(.)$ and $\cos(.)$ oscillate. Therefore the values of the components of the vector x_n oscillate.

c) Construct a subsequence of the sequence $\{\mathbf{x}_n\}$, defined in the previous part, that is Cauchy.

Ans: Make $\tau(n) = 4n$. Then note that
$\{x_{\tau(n)}\} = \{(-1)^{4n}\sin(\pi/2^{4n} + 2\pi n), 0, \cos(\pi/2^{4n} + 2\pi n)\}$
Note that $\cos(\theta) = \cos(\theta + 2\Pi k)$ and $\sin(\theta) = \sin(\theta + 2\Pi k)$ for any angle θ and integer k. Then
$\{x_{\tau(n)}\} = \{\sin(\pi/2^{2n}), 0, \cos(\pi/2^{2n})\}$
the sequence converges to (0,0,1).

d) Determine if S is an open or a closed set in $\mathbb{R}^3$

Ans: S is a sphere in $\mathbb{R}^3$ with radius 1 defined on spherical coordinates. S is a closed set in $\mathbb{R}^3$.

e) Construct a new set S' by adding a new parameter ρ to the definition of S. Give conditions on this new parameter so that S' is open but for which all points in S are boundary points.

Ans: $S' = \{(x, y, z)\,|x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi, \phi > 0, \theta > 0, \rho \geq 0\}$
If $\rho < 1$ then S' is open and all points in S are boundary points.

f) Is S a bounded set? Is S^c a bounded set?

Ans: S is bounded. S^c is not.

Problem 5: Hemi-continuity. (14 points)

a) Let $\psi : \mathbb{R}^2 \twoheadrightarrow \mathbb{R}^2$ be defined as: $\psi(x, y) = (z, w)$, where

$$z = \begin{cases} [-1, 1] & \text{if } x + y < 1 \\ 3/2 & \text{elsewhere} \end{cases} \quad \textit{and} \quad w = \begin{cases} [-1, 1] & \text{if } x + y < 1 \\ 2/3 & \text{elsewhere} \end{cases}$$

Is this correspondence upper hemi continuous? Lower hemi continuous? Continuous?

Ans: It is neither lhc nor uhc, and hence not continuous. To see why this correspondence is not uhc, let $\{x_n, y_n\} = \{\frac{1}{2} - \frac{1}{n}, \frac{1}{2} - \frac{1}{n}\}$, then $\{x_n, y_n\} \to \{\frac{1}{2}, \frac{1}{2}\}$. Let $\{z_n, w_n\} = \{\frac{1}{n}, \frac{1}{n}\}$. Then $\{z_n, w_n\} \to \{0, 0\}$. Clearly, the sequence $\{z_n, w_n\} \in \psi(x_n, y_n)$, but $(0, 0) \notin \psi(\frac{1}{2}, \frac{1}{2})$. To see why it is not lhc, note that a sequence in the interval [-1,1] can not converge to 3/2. Therefore it is not lhc.

b) Let $\phi : \mathbb{R}^2 \twoheadrightarrow \mathbb{R}^2$ be defined as: $\phi(x, y) = (z, w)$, where

$$z = \begin{cases} [-1, 1] & \text{if } x + y \geq 1 \\ 3/2 & \text{elsewhere} \end{cases} \quad \textit{and} \quad w = \begin{cases} [-1, 1] & \text{if } x + y > 1 \\ 2/3 & \text{elsewhere} \end{cases}$$

Is this correspondence upper hemi continuous? Lower hemi continuous? Continuous?

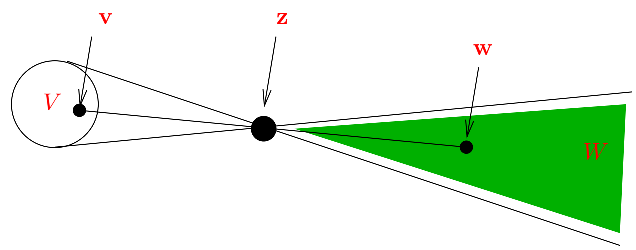

FIGURE 1. The set W is the shaded area on the right

Ans: It is neither. To see why it is not uhc, let $\{x_n, y_n\} = \{\frac{1}{2} + \frac{1}{n}, \frac{1}{2} + \frac{1}{n}\}$, then $\{x_n, y_n\} \to \{\frac{1}{2}, \frac{1}{2}\}$. Let $\{z_n, w_n\} = \{\frac{1}{n}, \frac{1}{n}\}$. The sequence $\{z_n, w_n\} \in \psi(x_n, y_n)$, but $(0,0) \notin \psi(\frac{1}{2}, \frac{1}{2})$ because $w(1,1) = 2/3$.
To see why it is not lhc, take $\{x_n, y_n\} = \{\frac{1}{2} - \frac{1}{n}, \frac{1}{2} - \frac{1}{n}\}$. $\{x_n, y_n\} \to \{\frac{1}{2}, \frac{1}{2}\}$. Since $z_n = 3/2$ for all x_n and y_n such that $x_n + y_n < 1$, then we can not construct a sequence $\{z_n, y_n\} \in \phi(x_n, y_n)$ so that $\{z_n, y_n\} = \{3/2, y_n\}$ that converges to $\{1, 1\}$. Therefore is not lhc.

c) Let $\chi : \mathbb{R}^2 \twoheadrightarrow \mathbb{R}^2$ be defined as: $\chi(x, y) = (z, w)$, where
$z = \begin{cases} 0 & \text{if } (x,y)\text{=(0,0)} \\ 1 & \text{elsewhere} \end{cases}$ *and* $w = \begin{cases} 0 & \text{if } (x,y)\text{=(0,0)} \\ -1 & \text{elsewhere} \end{cases}$
Is this correspondence upper hemi continuous? Lower hemi continuous? Continuous?

Ans: It is neither uhc nor lhc, To see why it is not uhc, let $\{x_n, y_n\} = \{(-1/n)^n, (-1/n)^n\}$, then $\{x_n, y_n\} \to \{0, 0\}$. Let $\{z_n, w_n\} = \{1, -1\}$. Then $\{z_n, w_n\} \in \chi(x_n, y_n)$ and $\{z_n, w_n\} \to \{1, -1\}$ but $\{1, -1\} \notin \chi(0, 0)$. To see why it is not lhc, take the point $(z*, w*) = (0, 0)$. You can not write a sequence x_n, y_n such that it converges to $(z*, w*)$ because $(x_n, y_n) \neq (0, 0)$ implies $(z_n, w_n = (-1, 1)$.

Problem 6: Strict Quasi convexity. (16 points)

a) (This is a little lemma that will help you prove the main result, below.)
Let V be an open set and fix $\mathbf{z} \in \mathbb{R}^n$. Let

$$W = \{\mathbf{w} \in \mathbb{R}^n : \mathbf{w} = \alpha\mathbf{z} + (1 - \alpha)\mathbf{v},\ \text{for some } \mathbf{v} \in V \text{ and } \alpha > 1\}.$$

(See Fig. 1). Prove that W is an open set.

Ans: Pick $\mathbf{w} \in W$. By assumption, $\mathbf{w} = \alpha\mathbf{z} + (1-\alpha)\mathbf{v}$, for some $\mathbf{v} \in V$ and $\alpha > 1$. Since V is open, there exists $\delta > 0$ such that $B(\mathbf{v}, \delta) \subset V$. We need to find $\epsilon > 0$ such that $B(\mathbf{w}, \epsilon) \subset W$. Let $\epsilon = \delta(\alpha - 1) > 0$ and note that for all $\mathbf{dx} \in B(0, \epsilon)$, $\mathbf{v} + \frac{\mathbf{dx}}{(\alpha-1)} = \mathbf{v} - \frac{\mathbf{dx}}{(1-\alpha)} \in B(\mathbf{v}, \delta) \subset V$.

It follows now from the definition of W that $\alpha\mathbf{z} + (1-\alpha)\left(\mathbf{v} - \frac{\mathbf{dx}}{(1-\alpha)}\right) = \mathbf{w} + \mathbf{dx} \in W$. We have established, then, that $B(\mathbf{w}, \epsilon) \subset W$, proving that W is open.

b) A continuous function $f : X \to \mathbb{R}$ is defined to be *strictly quasi-convex* if it satisfies the property (A), where

$$\forall \mathbf{x}, \mathbf{y} \in X \text{ s.t.} f(\mathbf{y}) \leq f(\mathbf{x}), \forall \lambda \in (0,1), f(\lambda\mathbf{y} + (1-\lambda)\mathbf{x}) < f(\mathbf{x}). \tag{A}$$

Show that condition (A) defining strict quasi-convexity is satisfied if and only if conditions (B) and (C) are satisfied, where

$$\text{every lower contour set of } f \text{ is a strictly convex set} \tag{B}$$

$$\text{every level set of } f \text{ has an empty interior} \tag{C}$$

Proving that (B) and (C) imply (A) is a bit tricky. So I suggest that you might consider trying to prove that $C \cap \neg A \implies \neg B$, by proceeding along the following lines.

i) Find a point $\mathbf{z}$ on the line segment joining $\mathbf{x}$ and $\mathbf{y}$, where $f(\mathbf{y}) \leq f(\mathbf{x})$, that violates condition (A).

ii) There are now two mutually exclusive possibilities
   (i) x belongs to the interior of the lower contour set corresponding to $f(\mathbf{x})$.
   (ii) x doesn't belong to the interior of the lower contour set corresponding to $f(\mathbf{x})$, violating condition B.

   If the latter case holds, then you are done. Assume therefore that the former case applies.

iii) Now apply property (C) to the point $\mathbf{z}$, use your answer to the first part of this question, plus the continuity of f, use property (C) a second time, and conclude that for some $\alpha < f(\mathbf{x})$, the lower contour set of f corresponding to α is not a convex set.

Ans: Proof that (A) $\implies$ (B) & (C): Assume that f is strictly quasi-convex. First, let S denote the lower contour set of f corresponding to $\alpha \in \mathbb{R}$ and pick $\mathbf{x}, \mathbf{y} \in S$. We need to establish that for $\lambda \in (0,1)$, $f(\lambda\mathbf{y}+(1-\lambda)\mathbf{x})$ belongs to the interior of S. Without loss of generality, assume that $f(\mathbf{y}) \leq f(\mathbf{x})$. By assumption, $f(\mathbf{x}) \leq \alpha$, so that by definition of strict quasi convexity, $f(\lambda\mathbf{y}+(1-\lambda)\mathbf{x}) < f(\mathbf{x}) \leq \alpha$. Since f is continuous, there exists a neighborhood U of $\lambda\mathbf{y}+(1-\lambda)\mathbf{x}$ such that for all $z \in U$, $f(z) < \alpha$. Hence $\lambda\mathbf{y} + (1-\lambda)\mathbf{x}$ is an interior point of S. We now need to establish that the level sets of f have empty interior. Let L be a level set of f, pick $\mathbf{x} \in L$ and pick $\epsilon > 0$. We need to show that $B(\mathbf{x}, \epsilon) \not\subseteq L$. Pick $\mathbf{y} \in B(\mathbf{x}, \epsilon)$, so that $f(\mathbf{y}) \leq f(\mathbf{x})$. Let $\mathbf{z} = (0.5\mathbf{x} + 0.5\mathbf{y})$. Clearly $\mathbf{z} \in B(\mathbf{x}, \epsilon)$. Since f is strictly quasi-convex, $f(\mathbf{z}) < f(\mathbf{x})$ and hence $\mathbf{z} \notin L$. We have established, then, that $B(\mathbf{x}, \epsilon) \not\subseteq L$.

Proof that (B) & (C) $\implies$ (A): Following the suggestion above, we will show that $C \cap \neg A \implies \neg B$. Since f is not strictly quasi-convex, there must exist $\mathbf{x}, \mathbf{y} \in X$ s.t.$f(\mathbf{y}) \leq f(\mathbf{x})$, $\lambda \in (0,1)$ such that $\mathbf{z} = \lambda\mathbf{y} + (1-\lambda)\mathbf{x}$ and $f(\mathbf{z}) \geq f(\mathbf{x})$. If $\mathbf{z}$ is *not* an interior point of the lower contour set of f corresponding to $f(\mathbf{x})$ then we are done. Assume, therefore, that $\mathbf{z}$ *is* an interior point of the lower contour set of f corresponding to $f(\mathbf{x})$. That is, $f(\mathbf{z}) = f(\mathbf{x})$ and there exists an open set U containing $\mathbf{z}$ such that for all $\mathbf{u} \in U$, $f(\mathbf{u}) \leq f(\mathbf{x})$. Since property (C) holds, z is not an interior point of the level set of f corresponding to $f(\mathbf{x})$. Hence there exists some point $\mathbf{u} \in U$ such that $f(\mathbf{u}) < f(\mathbf{x})$. Since f is continuous, u must belong to an open set $V \subset U$ such that for all $\mathbf{v} \in V$, $f(\mathbf{v}) < f(\mathbf{x})$. Now consider the set $W \cap U$, where $W = \{\mathbf{w} \in \mathbb{R}^n : \mathbf{w} = \alpha\mathbf{z} + (1-\alpha)\mathbf{v},$ for some $\mathbf{v} \in V$ and $\alpha > 1\}$. From part a) of this question, W is open and hence $W \cap U$ is open. Moroever, $W \cap U$ belongs to the lower contour set of f corresponding to $f(\mathbf{x})$. From property (C), therefore, it cannot be the case that $f(\mathbf{w}) = f(\mathbf{x})$ for all $\mathbf{w} \in W \cap U$, i.e., there exists $\mathbf{w} \in W \cap U$ such that $f(\mathbf{w}) < f(\mathbf{x})$. By definition of W, there exists $\mathbf{v} \in V$ such that $\mathbf{z}$ lies on the line segment joining $\mathbf{v}$ and $\mathbf{w}$. Let $\alpha = \max(f(\mathbf{v}), f(\mathbf{w})) < f(\mathbf{x})$. Obviously, $\mathbf{v}$ and $\mathbf{w}$ belong to the lower contour set of f corresponding to α. However, since $f(\mathbf{z}) = f(\mathbf{x})$, $\mathbf{z}$ does *not* belong to this lower contour set. Hence the lower contour set of f corresponding to f is not a convex set. We have shown, then that property B above is violated.

Problem 7: Kuhn Tucker. (14 points)

For $c \in \mathbb{R}_{++}$, consider the problem

$$\begin{aligned} &\max x^{1/2}y^{1/2} \\ x^2 + y^2 &= 1 && (1) \\ cx + \sqrt{2}y &\leq 2 && (2) \\ x &\geq 0 && (3) \\ y &\geq 0 && (4) \end{aligned}$$

a) Write this problem in the standard format, i.e., $\max f(\mathbf{x})$ s.t. $\mathbf{g}(\mathbf{x}) \leq \mathbf{b}$.

Ans:

$$\begin{aligned} &\max x^{1/2}y^{1/2} \\ x^2 + y^2 &\leq 1 && (1\leq) \\ -x^2 - y^2 &\leq -1 && (1\geq) \\ cx + \sqrt{2}y &\leq 2 && (2) \\ -x &\leq 0 && (3) \\ -y &\leq 0 && (4) \end{aligned}$$

b) Identify conditions on c, if such conditions exist, such that
   i) the constraint (2) is neither binding nor satisfied with equality at the solution to the problem.
   ii) the constraint (2) is satisfied with equality at the solution to the problem but is not binding
   iii) the constraint (2) is binding.

   In each of these cases graph the problem, *clearly* indicating the both the constraint set and the solution to the problem. Then provide the analytic solution. In each case, specify which of the other constraints are binding.

Ans: See Fig. 2. Constraints (3) and (4) are slack for all values of $c \in \mathbb{R}$. For the other constraints, there are three regions to consider

i) when $c < \sqrt{2}$: constraint $(1\leq)$ is binding and constraints $(1\geq)$ and (2) are slack. (If $(1\leq)$ were relaxed, the objective would be increased by increasing both x and y.)

ii) when $c > \sqrt{2}$, constraint $(1\leq)$ is slack and constraints $(1\geq)$ and (2) are binding. (To see this, note that if $(1\geq)$ were relaxed, creating a little band below the circle of radius 1., the objective would be increased by sliding down the straight line, i.e., increasing x and reducing y. On the other hand, if $(1\leq)$ were relaxed, creating a band above the circle of radius 1, the optimum values of x and y would be unchanged.)

iii) when $c = \sqrt{2}$, constraints $(1\leq)$, $(1\geq)$ and (2) are all satisfied with equality, but technically, *none* of them are binding: if you relax any one of them without relaxing the others, the solution stays fixed.

The analytic solution provided here is relatively un-anal in nature. I.e., I'm going to work from the pictures and not even write down the constraints that are slack. The gradient of the objective is $\frac{\sqrt{xy}}{2}(1/y, 1/x)$ When $c \leq \sqrt{2}$, this vector can be written as a scalar multiple of the gradient of constraint $(1\leq)$, i.e., $2(x, y)$ that is,

$$\frac{\sqrt{xy}}{4} = (x^2, y^2)$$

Hence $x = y$ and from $(1\leq)$ and $(1\geq)$, this means that $x = y = \sqrt{2}/2$. When $c > \sqrt{2}$, the gradient of the objective is equal to $\lambda_2(c, \sqrt{2}) - 2\lambda_1(x, y)$, for some $(\lambda_1, \lambda_2) \gg 0$. Also, since constraints $(1\leq)$ and (2) are satisfied with equality and y is nonnegative, we have $y = (2 - cx)/\sqrt{2}$ and $x = \sqrt{1 - y^2}$. Combining these equations and applying the quadratic formula, we get that $y = (2\sqrt{2} \pm c\sqrt{c^2 - 2})/(2 + c^2)$. From Fig. 2, it's clear that the larger of the two roots yields a higher value of the objective, so that the optimal value of y is $y = (2\sqrt{2} + c\sqrt{c^2 - 2})/(2 + c^2)$. Since constraint 2 holds, it follows that $x = (2 - \sqrt{2}y)/c = (c - \sqrt{c^2/2 - 1})/(1 + c^2/2)$.

c) For the case in which constraint (2) is binding, indicate how the degree of bindingness (i.e., the magnitude of the Lagrangian multiplier) changes as c changes. Ideally, support your answer graphically.

Ans: As c increases, both of the binding constraints become *less* binding. Fig. 3 illustrates what's going on. It's easy to see from the figure why constraint (2) becomes less binding: when $c \approx \sqrt{2}$, the gradient of the second constraint points in essentially the same direction as the gradient of the objective. As c increases, the gradient of the objective moves "toward the middle" of the cone generated by the gradients of the two constraints, so the weight on constraint (2) decreases. It's much less easy to give an intuitive explanation for why constraint $(1\geq)$ becomes less binding. Fig. 4 plots the two lagrangians as c increases from $\sqrt{2}$ to 10.

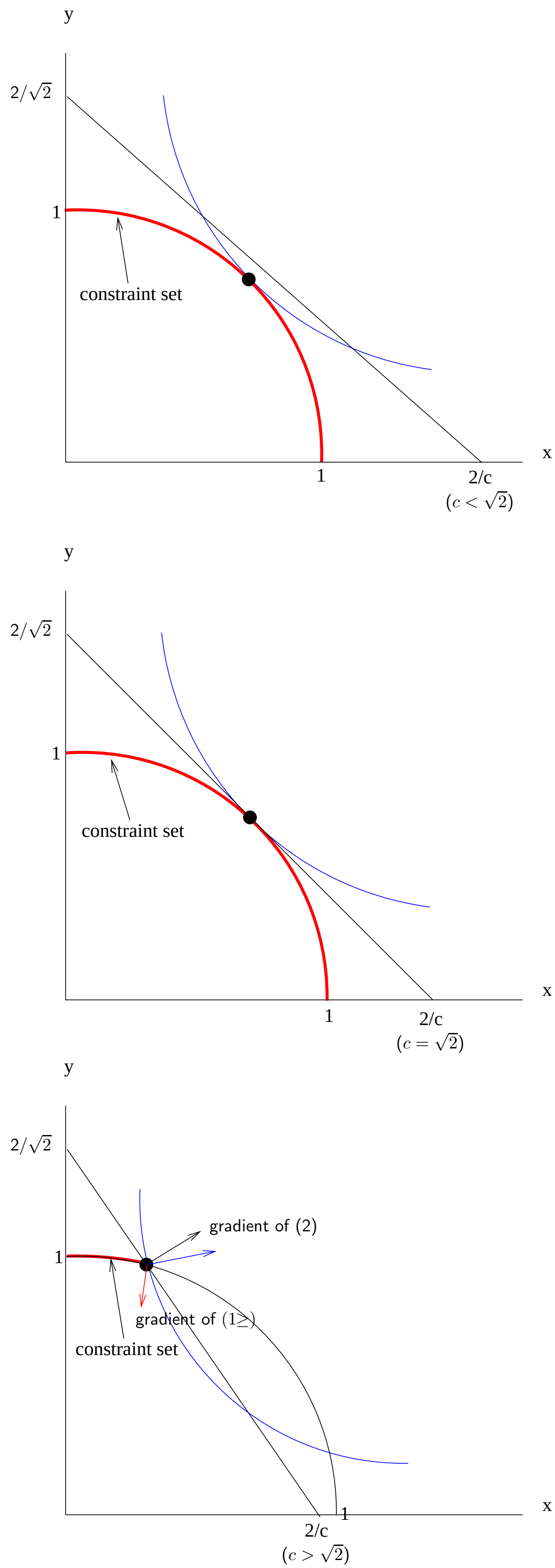

Figure 2. Graphical solution to problem 7

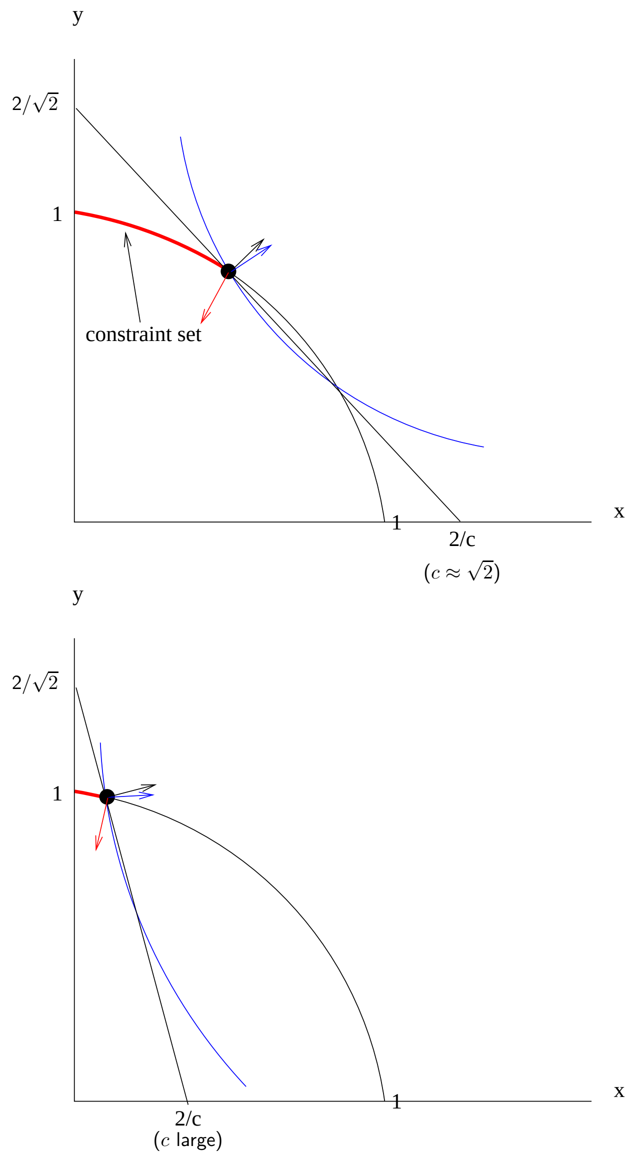

FIGURE 3. How the Lagrangians change as c increases

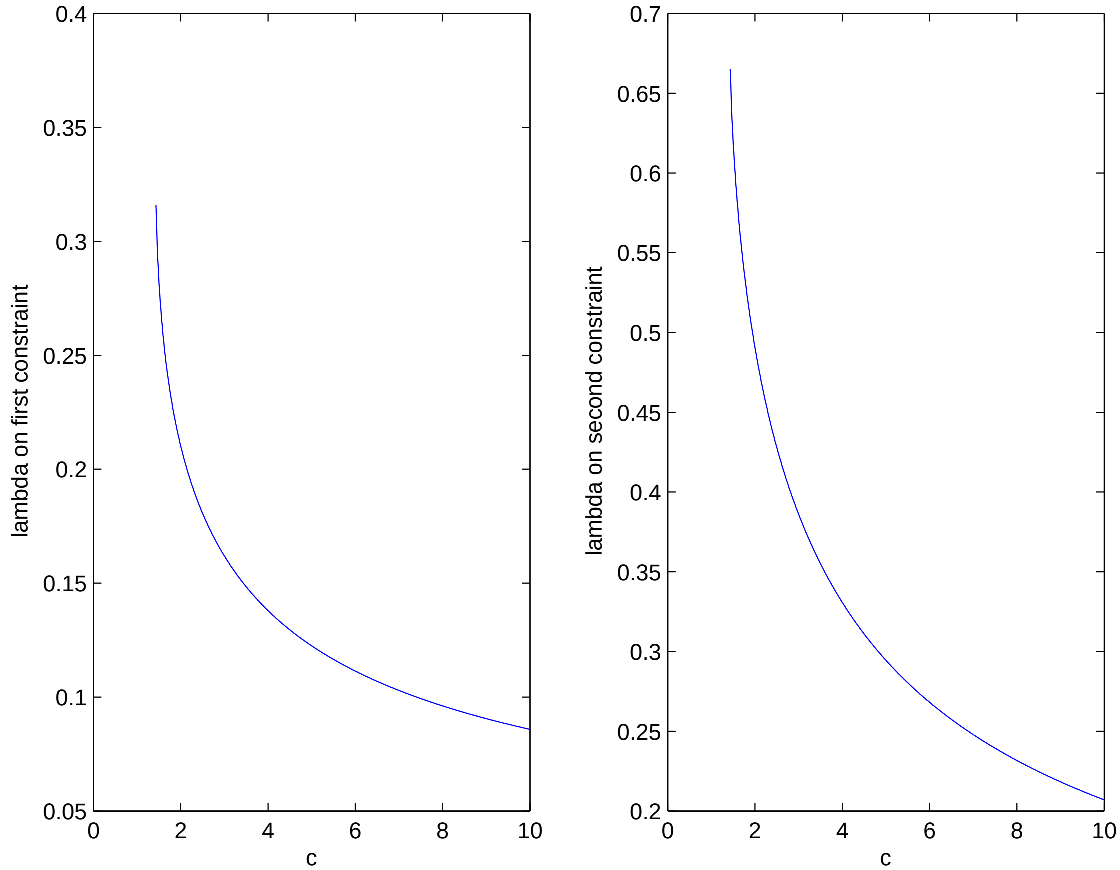

FIGURE 4. How the lagrangians change as c increases