Final Exam - Answer key

Problem 1 (Real Analysis) [36 points]:

A) [*6 points*] Show that if $\{x_n\}$ is a convergent sequence, then for any N, the sequence given by the average $y_n = \frac{x_{n+1}+x_{n+2}+...+x_{n+N}}{N}$ converges to the same limit. Is it possible for the sequence $\{y_n\}$ to converge even if $\{x_n\}$ does not?

Ans: Suppose $\{x_n\}$ converges to x. Then y_n converges to $N\frac{x}{N} = x$. Now let $x_n = -1^n$ and let $N = 2$. Clearly $\{x_n\} = \{-1, 1, -1, 1, ...\}$ does not converge but $y_n = \{0, 0, 0, 0, ...\}$ does.

B) [*6 points*] Show that if K is a compact subset of $\mathbb{R}$ and F is a closed subset of $\mathbb{R}$, then $K \cap F$ is compact.

Ans: Since K is compact, then there exists $\underline{b}, \bar{b} \in \mathbb{R}$ such that $\underline{b}$ is a lower bound, and $\bar{b}$ is an upper bound, for K. Since $K \cap F \subset K$, $\underline{b}$ and $\bar{b}$ are necessarily lower and upper bounds for $K \cap F$ as well. Hence $K \cap F$ is bounded. Moreover the intersection of closed sets is closed, so $K \cap F$ is closed. $K \cap F$ is thus closed and bounded, and hence compact.

C) [*7 points*] Prove that every finite set in $\mathbb{R}$ is compact.

Ans: Let X be a finite subset of $\mathbb{R}$. Since X is finite, it has a maximum and a minimum. The maximum is an upper bound for X; the minimum is a lower bound for X; hence X is bounded. For each distinct pair $x, y \in X$, let $\epsilon^{x,y}$ denote the euclidean distance between x and y. Since X is finite, $\{\epsilon^{x,y} : x, y \in X\}$ is a finite set of positive numbers. Let $\underline{\epsilon} > 0$ be the minimum of this set. Now note that no point in X is an accumulation point, since for $x \in X$, $y \in B(x, \underline{\epsilon}/2)$ implies $y = x$. A set is closed if it contains all of its accumulation points. Since X has no accumulation points, this requirement is satisfied vacuously.

D) [*7 points*] Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences in $\mathbb{R}$. Show that if $x_n \leq y_n \leq z_n$ for all n, and if $lim\{x_n\} = lim\{z_n\} = \ell$ then $lim\{y_n\} = \ell$ as well.

Ans: Fix $\epsilon > 0$ and $N \in \mathbb{N}$ such that for all $n > N$, $x_n, z_n \in B(\ell, \epsilon)$. Fix $n > N$. Since $y_n \leq z_n$, $y_n < \ell + \epsilon$. Since $y_n \geq x_n$, $y_n > \ell - \epsilon$. Hence $y_n \in B(\ell, \epsilon)$. We have established then that for all $n > N$, $y_n \in B(\ell, \epsilon)$. Hence $\{y_n\}$ converges to ℓ.

E) [*10 points*] Show that the empty set and $\mathbb{R}^n$ are both open and closed in $\mathbb{R}^n$. Prove that no other subsets of the $\mathbb{R}^n$ can be both open and closed.

Ans: A set X is open if for any point $x \in X$, there exists $\epsilon > 0$ and a ball $B(x, \epsilon|X) \subset X$. Since there exists no x contained in the empty set, this definition is satisfied trivially for that set. Hence the empty set is open in $\mathbb{R}^n$. Now consider $\mathbb{R}^n$. For any point $x \in \mathbb{R}^n$ and any $\epsilon > 0$, the ball $B(x, \epsilon, \mathbb{R}^n)$ is necessarily contained in $\mathbb{R}^n$. Hence $\mathbb{R}^n$ is open in $\mathbb{R}^n$. A set is closed if its complement is open. The empty set and $\mathbb{R}^n$ are complements of each other in $\mathbb{R}^n$. Hence they are both closed as well. Now consider a set $X \in \mathbb{R}^n$ which is neither empty or $\mathbb{R}^n$ itself, i.e. both X and X^c (the complement of X) are nonempty. Let x be an boundary point of X, i.e., a point x such that for all $\epsilon > 0$, there exists $y, z \in B(x, \epsilon|\mathbb{R}^n)$ such that $y \in X$ and $z \in X^c$.

If X is closed, then $x \in X$. But in this case, there does not exist an open ball around x that is contained in X. So X cannot be open.

Problem 2 (Linear Algebra) [36 points]:

A) [*3 points*] Give necessary and sufficient conditions for an $n \times m$ matrix A to be invertible.

Ans: $n = m$ and $det(A) \neq 0$.

B) [*3 points*] Verify that $\mathbf{v}^1 = \begin{bmatrix} 0.5 \\ \sqrt{3/4} \end{bmatrix}$ is a unit eigenvector of the matrix $A = \begin{bmatrix} 5 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{bmatrix}$.

Ans: First note that $||\mathbf{v}^1|| = \sqrt{0.25 + 0.75} = 1$ so that $\mathbf{v}^1$ is indeed a unit vector. Second $A\mathbf{v} = [1, \sqrt{3}] = 2[0.5, \sqrt{3/4}]$, so that $A\mathbf{v} = \lambda\mathbf{v}$, for $\lambda = 2$.

C) [*3 points*] What is the eigenvalue corresponding to $\mathbf{v}^1$. Verify.

Ans: 2. I just verified this.

D) [*3 points*] Identify a second, distinct unit eigenvector $\mathbf{v}^2$ of the matrix A.

Ans: Since there necessarily exist pairwise orthogonal eigen-vectors, we can solve for $\begin{bmatrix} 0.5 & \sqrt{3/4} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ x \end{bmatrix} = 0$. i.e., $x = 0.5\sqrt{4/3} = -\sqrt{1/3}$. Now $\begin{bmatrix} 1 \\ -\sqrt{1/3} \end{bmatrix}$ is an eigenvector but it is not a unit eigenvector. To get the corresponding unit eigenvector, we divide by the norm, i.e., $\sqrt{4/3}$ to obtain $\mathbf{v}^2 = \begin{bmatrix} \sqrt{3/4} \\ -\sqrt{1/4} \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 \\ -1/2 \end{bmatrix}$.

E) [*3 points*] What is the eigenvalue corresponding to $\mathbf{v}^2$. Verify.

Ans: The second eigenalue is 6. To verify this, note that $A\mathbf{v}^2 = \begin{bmatrix} 6\sqrt{3}/2 \\ -3 \end{bmatrix} = 6 \begin{bmatrix} \sqrt{3}/2 \\ -1/2 \end{bmatrix} = 6\mathbf{v}^2$.

F) [*3 points*] What can you say about the definiteness or otherwise of A?

Ans: Since both eigenvalues are positive, the matrix is positive definite.

G) [*5 points*] Prove that a 2×2 matrix A that has less than full rank is semi-definite.

Ans: A 2×2 matrix that has less than full rank either has two zero eigenvalues or one non-zero eigenvalue. In the former case, $\mathbf{x}'A\mathbf{x} = 0$, for all $\mathbf{x}$ so that the matrix is both positive and negative semi-definite. In the latter case, if the non-zero eigenvalue is positive, then $\mathbf{x}'A\mathbf{x} \geq 0$, for all $\mathbf{x}$, i.e., the matrix is positive semi-definite; if the non-zero eigenvalue is negative, then $\mathbf{x}'A\mathbf{x} \leq 0$, for all $\mathbf{x}$, i.e., the matrix is negative semi-definite.

H) [*3 points*] Now construct a matrix B with the following properties:

- all of its elements are non-zero
- it is symmetric
- all but one of its elements are equal to the corresponding elements of A
- it is semi-definite

Ans: $B = \begin{bmatrix} 5 & -\sqrt{3} \\ -\sqrt{3} & 3/5 \end{bmatrix}$. The first three of the required properties are obviously satisfied. To check that it's semi-definite, note that its determinant is $3 - 3 = 0$, so that it has less than full rank. From part G), the matrix is therefore semi-definite.

I) [*5 points*] Is the matrix you have constructed positive or negative semi-definite?

Ans: It's positive definite. To verify this note that its one non-zero eigenvector, $\begin{bmatrix} 5 \\ -\sqrt{3} \end{bmatrix}$ has a positive eigenvalue; For ease of computation, consider the non-unit eigenvector $\mathbf{v} = \begin{bmatrix} 1 \\ -\sqrt{3}/5 \end{bmatrix}$ and note that $B\mathbf{v} = \begin{bmatrix} 28/5 \\ -\sqrt{3} \times 28/5 \end{bmatrix} = 28/5 \begin{bmatrix} 1 \\ -\sqrt{3}/5 \end{bmatrix}$. Thus the one non-zero eigenvalue is positive.

J) [*5 points*] For $n > 2$, an $n \times n$ matrix that has less than full rank need not be semi-definite. Explain why not.

Ans: For $n > 2$, a matrix can have at least one eigenvalue, at least one positive eigenvalue and at least one negative eigenvalue. In this case it will be indefinite.

Problem 3 (Calculus) [36 points]:

You operate a factory that makes cars according to a production function, $F(K, L) = 3K^{1/3}L^{2/3}$.

A) [*5 points*] How many cars do you produce when your input mix is $(27, 125)$?

Ans: $F(27, 125) = 3 \times 3 \times 25 = 225$.

For the next four parts, use the answer you obtained in part A).

B) [*5 points*] Use the differential to compute a first order approximation to the number of cars you would produce if your input mix input were $(36, 130)$?

Ans: $\triangledown F(K, L)$ is $F(K, L)/3 \times (1/K, 2/L)$. So

$$\begin{aligned} F(36,130) &\approx F(27,125) + \triangledown F(27,125) \cdot (9,5) = 225 + 75 \times (1/27, 2/125) \cdot (9,5) \\ &= 225 + 75 \times (1/3 + 2/25) = 225 + 25 + 6 = 256 \end{aligned}$$

C) [*5 points*] What is the directional derivative of F $(27, 125)$ in the direction $(9, 5)$?

Ans:

$$F_{9,5} = \bigtriangledown F(27,125)\cdot(9/\sqrt{81+25}, 5/\sqrt{81+25}) = 75\times(1/27+2/125)\cdot(9,5)/\sqrt{81+25}$$
$$= 75\times(1/3+10/125)/\sqrt{106}$$
$$= (25+6)/\sqrt{106} = 31/\sqrt{106}$$

D) [*5 points*] Using the answer you obtained in part C), compute a first order approximation to the number of cars you would produce if your input mix input were $(36,130)$?

Ans:

$$F(30,130) \approx F(27,125)+F_{9,5}||(9,5)|| = 225+31/\sqrt{106}\times\sqrt{106}=256$$

E) [*6 points*] Compute a *second* order approximation to the number of cars you would produce if your input mix were $(36,130)$? (Since you don't have calculators, we'll give you full credit for this part even if you don't complete the numerical computation. But you should go up to the point at which a calculator would be necessary.)

Ans: The Hessian of F is $H = 2/9F\begin{bmatrix}-1/K^2 & 1/(LK)\\ 1/(LK) & -1/L^2\end{bmatrix}$. Evaluated at $(27,125)$ the Hessian is 50. The second order term in the Taylor expansion is $0.5[9,5]\times H\times[9,5]' = -4.30$. Thus, our second order approximation is $256-2.15=253.85$.

F) [*5 points*] In what proportions should you add K and L to $(27,125)$ if you want to increase production most rapidly?

Ans: Add in the proportions given by the gradient, i.e., $(1/27,2/125)/(1/27+2/125)$.

Your car company is unprofitable, so, in order to receive a government bailout, you must implement a new technology. Under this technology, your production inputs, K and L, are functions of time, t, and the interest rate, r. Specifically, $K(t,r) = 9\frac{t^2}{r}$ and $L(t,r)=t^2+r$ (note: your production function, F, does not change).

G) [*5 points*] Calculate the rate of change of output with respect to t when $t=10$ and $r=0.1$.

Ans: The rate of change is

$$\frac{dF}{dt} = F/3(1/K,2/L)\cdot(2t/r,2t) = 75\times(1/27,2/125)\cdot(18,20)$$
$$= 75\times(2/3+40/125) = 50+24 = 74.$$

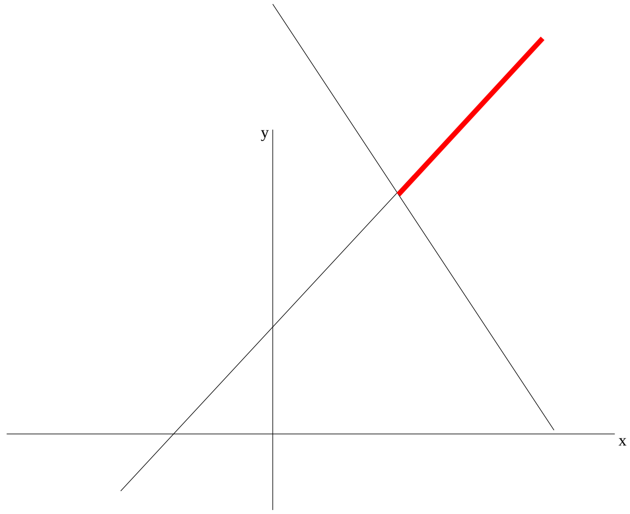

FIGURE 1. The constraint set in problem 4

Problem 4 (Kuhn Tucker) [36 points]:

Consider the problem

$$min_{x,y} \quad -3(x-10)(y-25)+(x-10)^3 \quad \text{s.t.} \quad 2x-y=-5, \quad 5x+2y \geq 37, \quad x \geq 0, \quad y \geq 0.$$

A) [*3 points*] Write this problem in the canonical NPP format.

Ans: The problem in canonical NPP format is

$$\max_{x,y} \quad 3(x-10)(y-25)-(x-10)^3 \quad \text{s.t.}$$

$$\begin{aligned} 2x-y &\leq -5 \\ y-2x &\geq 5 \\ -5x-2y &\leq -37 \\ -x &\leq 0 \\ -y &\leq 0 \end{aligned}$$

B) [*3 points*] Sketch the constraint set.

Ans: The constraint set is the thick red line in Fig. 1

C) [*3 points*] State what it means for the constraint qualification to be satisfied *in language that involves no mathematical symbols.*

Ans: The constraint qualification is satisfied at a point if the linearized version of the constraint set is, in a neighborhood of that point, almost identical to the original constraint set.

D) [*3 points*] What points in the constraint set satisfy the constraint qualification?

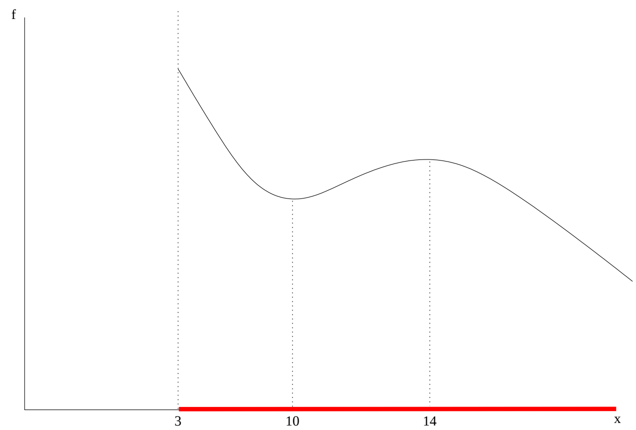

FIGURE 2. Graph of f

Ans: Since the constraints are already linear, the linearized version of the constraint set is, vacuously, identical to the constraint set at every point in the constraint set.

At this point, to facilitate solving the problem, convert it into a problem with *one* unknown, x, and *one* constraint. Call the objective function f.

E) [*4 points*] Write down the single variable constrained optimization problem.

Ans: A necessary condition for a solution to this problem is that $y = 2x + 5$. Rewriting the objective function as a function, f, of one variable, x, we have $f(x) = 3(x-10)(2x-20) - (x-10)^3 = 6(x-10)^2 - (x-10)^3 = (16-x)(x-10)^2$. Rewriting the second constraint in terms of x, we have $5x + 2(2x+5) \geq 37$, or $9x \geq 27$ or $x \geq 3$. Thus the single variable constrained optimization problem can be written as

$$\max_x \; 6(x-10)^2 - (x-10)^3 \quad \text{s.t.} \quad x \geq 3$$

F) [*4 points*] Write down the first and second derivatives of f.

Ans: $f'(x) = 12(x-10) - 3(x-10)^2 = 3(x-10)(4-(x-10)) = -3(x-10)(x-14) = -3(x^2 - 24x + 140)$ while $f''(x) = 6(12-x)$.

G) [*4 points*] Using your answer to part F), sketch the constraint set and objective function.

Ans: See Fig. 2. The thick red line is the constraint set.

H) [*4 points*] What values of x satisfy the KKT conditions for this problem? (Get help from your graphical answer in part G).)

Ans: Clearly, $f'(x) = 0$ at 10 and 14. Moreover, $f'(x) < 0$ at the lower bound on the constraint set $x = 3$. So all three points satisfy the KKT.

I) [*4 points*] What point on the constraint set solves the NPP? (Get help from your graphical answer in part G).)

Ans: From part F), $f''(x)$ is positive at 10 and negative at 14. So $x = 10$ is not a solution, but $x = 3$ and $x = 14$ are candidates. $f(14) = (16-14)(14-10)^2 = 32$; $f(3) = (16-3)(3-10)^2 = 637$. So 3 solves the NPP.

J) [*4 points*] Comment on why there are more than one values of x that solve the KKT.

Ans: The second order sufficiency conditions are not globally satisfied: Specifically, f is not quasi-concave and certainly not pseudo-concave.

Problem 5 (Comparative Statics) [36 points]:

A) Consider the function $f(x, y, \gamma) = xy + \gamma y$ subject to the following constraints: $g(x, y, \gamma) \leq 1$, $x \geq 0$, $y \geq 0$, where $g(x, y) = x^2 + \gamma y$.

(a) [*10 points*] For $\gamma = 1$, solve this maxmization problem using either the Lagrangian or KKT method.

Ans: The KKT conditions are

$$\begin{bmatrix} y & x+\gamma \end{bmatrix} = \lambda \begin{bmatrix} 2x & \gamma \end{bmatrix} \tag{1}$$

We'll try to solve this assuming that the first constraint is binding and the nonnegativity constraints are slack. In this case, $y = 1 - x^2$, so that when $\gamma = 1$, the KKT conditions become

$$\begin{bmatrix} 1-x^2 & x+1 \end{bmatrix} = \lambda \begin{bmatrix} 2x & 1 \end{bmatrix}$$

Solving the second equation, we obtain $\lambda = x + 1$. Substituting into the first, we get $1 - x^2 = (x + 1)2x$ or $3x^2 + 2x - 1 = 0$ or $(3x - 1)(x + 1) = 0$. The unique positive solution to this equation is $x = 1/3$, hence $y = 8/9$, $\lambda = 4/3$. Double-checking the KKT conditions for arithmetic errors, the l.h.s. of (1) is $\begin{bmatrix} 8/9 & 4/3 \end{bmatrix}$ while the r.h.s. is $4/3 \begin{bmatrix} 2/3 & 1 \end{bmatrix} = \begin{bmatrix} 8/9 & 4/3 \end{bmatrix}$. We've established, then, that the KKT conditions are indeed satisfied at $(x^*, y^*, \lambda^*) = (1/3, 8/9, 4/3)$.

(b) [*10 points*] Now, use the envelope theorem to estimate the maximized value of f when $\gamma = 1.2$

Ans: To estimate the required value, we will use a first order Taylor expansion, i..e, $f(x^*(1), y^*(1), 1) + \frac{df(x^*(1),y^*(1),1)}{d\gamma}d\gamma$, where $d\gamma = 0.2$. By the envelope theorem, $\frac{df(x^*(\gamma),y^*(\gamma),\gamma)}{d\gamma} = \frac{\partial f(x^*(\gamma),y^*(\gamma)),\gamma)}{\partial\gamma} + \lambda^*(\gamma)\frac{\partial g(x^*(\gamma),y^*(\gamma)),\gamma)}{\partial\gamma}$. Now $\frac{\partial f(\cdot,\cdot,\cdot)}{\partial\gamma} = \frac{\partial g(\cdot,\cdot,\cdot)}{\partial\gamma} = y$, so that $\frac{df(x^*(\gamma),y^*(\gamma),\gamma)}{d\gamma} = y^*(\gamma)(1 - \lambda^*(\gamma))$. Plugging in the solution values we have just obtained, we have $f(x^*(1), y^*(1), 1) = 8/9 \times (1 + 1/3) = 32/27$ while $\frac{df(x^*(\gamma),y^*(\gamma),\gamma)}{d\gamma} = y^*(\gamma)(1 - \lambda^*(\gamma)) = 8/9 \times -1/3 = -8/27$. Hence, our first order approximation to $f(x^*(1.2), y^*(1.2), 1.2)$ is $32/27 - 0.2 \times 8/27 = 152/135$.

B) [*16 points*] Consider the problem $\max_x \; f(x; \alpha)$ s.t. $g(x; \alpha) \leq b$, where $x, \alpha, b \in \mathbb{R}$, f and g are twice continuously differentiable, $g_x(\cdot, \alpha) > 0$. Let $x^*(\alpha)$ denote the solution to this problem, given α. Use the implicit function theorem to identify sufficient conditions for $x^*(\cdot)$ to be everywhere strictly increasing in α. Are the conditions you identified necessary as well? If so prove it. If not, provide a counter-example.

Ans: The Lagrangian for this problem is $L(x, \lambda; \alpha) = f(x; \alpha) + \lambda(b - g(x, \alpha))$. To determine the relationship between x and α we apply the implicit function theorem to the zero level set of the first order conditions of the Lagrangian. We have

$$\begin{aligned} L_x &= 0 = f_x(x, \alpha) - \lambda g_x(x, \alpha) \\ L_\lambda &= 0 = b - g(x, \alpha) \end{aligned}$$

Applying the implicit function theorem to these conditions, we have

$$\begin{bmatrix} L_{x,x} & L_{x,\lambda} \\ L_{\lambda,x} & L_{\lambda,\lambda} \end{bmatrix} = \begin{bmatrix} (f_{xx} - \lambda g_{xx}) & -g_x \\ -g_x & 0 \end{bmatrix}$$

while

$$\begin{bmatrix} L_{x,\alpha} \\ L_{\lambda,\alpha} \end{bmatrix} = \begin{bmatrix} (f_{x,\alpha} - \lambda g_{x,\alpha}) \\ -g_\alpha \end{bmatrix}$$

and

$$\begin{bmatrix} dx/d\alpha \\ d\lambda/d\alpha \end{bmatrix} = -\begin{bmatrix} L_{x,x} & L_{x,\lambda} \\ L_{\lambda,x} & L_{\lambda,\lambda} \end{bmatrix}^{-1} \begin{bmatrix} L_{x,\alpha} \\ L_{\lambda,\alpha} \end{bmatrix}$$

We can now apply Cramer's rule to obtain

$$\begin{aligned} dx/d\alpha &= -\det\left(\begin{bmatrix} L_{x,\alpha} & L_{x,\lambda} \\ L_{\lambda,\alpha} & L_{\lambda,\lambda} \end{bmatrix}\right) \Big/ \det\left(\begin{bmatrix} L_{x,x} & L_{x,\lambda} \\ L_{\lambda,x} & L_{\lambda,\lambda} \end{bmatrix}\right) \\ &= -\det\left(\begin{bmatrix} (f_{x,\alpha} - \lambda g_{x,\alpha}) & -g_x \\ -g_\alpha & 0 \end{bmatrix}\right) \Big/ -g_x^2 \\ &= -\left(g_x g_\alpha / g_x^2\right) = -\left(g_\alpha / g_x\right) \end{aligned}$$

Since g_x is positive by assumption, it follows that $dx/d\alpha$ will be positive iff $g_\alpha < 0$.

This condition is not necessary however for $x^*(\cdot)$ to be strictly increasing in α. For example, let $f(x; \alpha) = x$, and let $g(x; \alpha) = x - \alpha^3$. Our maximization problem is now $\max_x\ x$ s.t. $x \leq b + \alpha^3$. Clearly the solution to this problem is globally strictly increasing in α. However, when $\alpha = 0$, then $g_\alpha = 0$. Hence the condition $g_\alpha < 0$ is not necessary for $x^*(\cdot)$ to be strictly increasing in α.