Fall2007 **ARE211**

Final Exam - Answer key

This is the final exam for ARE211. As announced earlier, this is an open-book exam. However, use of computers, calculators, Palm Pilots, cell phones, Blackberries and other non-human aids is forbidden.

Read all questions carefully before starting the test.

Allocate your 180 minutes in this exam wisely. The exam has 180 points, so aim for an *average* of 1 minute per point. However, some questions & parts are distinctly easier than others. Make sure that you first do all the easy parts, before you move onto the hard parts. Always bear in mind that if you leave a part-question completely blank, you cannot conceivably get any marks for that part. The questions are designed so that, to some extent, even if you cannot answer some parts, you will still be able to answer later parts. Even if you are unable to show a result, you are allowed to use the result in subsequent parts of the question. So don't hesitate to leave a part out. You don't have to answer questions and parts of questions in the order that they appear on the exam, *provided that you clearly indicate the question/part-question you are answering.* Finally, *always* keep in mind the famous maxim KISS (keep it simple, stupid).

Problem 1 [20 points]
Fix some metric d that applies to all parts of the following question. A function f is *uniformly continuous* on X if for all $\epsilon > 0$, $\exists \delta > 0$ such that for all $x, x' \in X$, $d(x, x') < \delta$ implies $d(f(x), f(x')) < \epsilon$. Now let $\{x_n\}$ be a Cauchy sequence in $X \subset \mathbb{R}$, let $f : X \to \mathbb{R}$ be some function, and consider the sequence $\{y_n\}$ defined by, for all n, $y^n = f(x^n)$.

A) [*10 points*] If f is continuous on X, is y^n Cauchy? If so, prove it. If not provide a counter example. (Your counter-example must specify the function f, its domain X and a Cauchy sequence in X.)

No. Counter example is $f(x) = 1/x$, defined on $X = R_{++}$; Cauchy sequence is $x_n = 1/n$. In this case, $f(x_n) = f(1/n) = n$, which is clearly not Cauchy.

B) [*10 points*] If f is uniformly continuous on X, is y^n Cauchy? If so, prove it. If not provide a counter example. (Your counter-example must specify the function f, its domain X and a Cauchy sequence in X.)

Yes. Fix $\epsilon > 0$. We need to show that there exists $N \in \mathbb{N}$, such that $m, n > N$ implies $d(y_n, y_m) < \epsilon$. Since f is uniformly continuous, there exists $\delta > 0$ such that $d(x, x') < \delta$ implies $d(f(x), f(y)) < \epsilon$. Since $\{x_n\}$ is Cauchy, there exists N such that $m, n > N$ implies $d(x_n, x_m) < \delta$ which in turn implies $d(f(x_n), f(x_m)) < \epsilon$.

Problem 2 [40 points]

Fix a vector $\mathbf{v}^0 \in \mathbb{R}^n$, two natural numbers $J > 1$ and $K > 1$, and a nonempty set $Q \subset \{1, ...JK\}$. Now let $\mathbb{M}$ denote the set of all $n \times JK$ matrices M such that $M = [\mathbf{x}^1, ..., \mathbf{x}^{JK}]$ where

$$\mathbf{x}^m = \begin{cases} \mathbf{v}^0 & \text{if } m \in Q \\ \mathbf{v}^k & \text{if } m \notin Q \text{ and } m = k + jK, \text{ for } j \in \{1, ..., J\} \text{ and } k \in \{1, ..., K\} \end{cases}$$

for some $n \times K$ matrix $V = [\mathbf{v}^1, ..., \mathbf{v}^K]$. (Note that each distinct element of $\mathbb{M}$ is defined by a *different* vector V.)

A) [*7 points*] Think of an example of an element of $\mathbb{M}$, for $K = 3$, $J = 4$ a set Q that has at least 3 elements and a $n \times K$ matrix V. Remember KISS! Write down Q, n, $\mathbf{v}^0$ and V Then write down your matrix M.

$n = 1$; $Q = \{2, 3, 4\}$; $\mathbf{v}^0 = 4$. $M = [1, 4, 4, 4, 2, 3, 1, 2, 3, 1, 2, 3]$.

B) [*4 points*] Identify conditions under which $\mathbb{M}$ is a vector space.

The condition is that $\mathbf{v} = 0$.

C) [*11 points*] Demonstrate that if the conditions you identified in part B) are satisfied, $\mathbb{M}$ is indeed a vector space.

Consider $M^1, M^2 \in \mathbb{M}$ and $\alpha \in \mathbb{R}^2$.
Let $M^3 = \alpha_1 M^1 + \alpha_2 M^2$ where for $i = 1, 2, 3$, $M^i = [\mathbf{x}^{1,i}, ..., \mathbf{x}^{JK,i}]$ and

$$\mathbf{x}^{i,3} = \begin{cases} 0 & \text{if } m \in Q \\ \alpha_1 \mathbf{x}^{1,k} + \alpha_2 \mathbf{x}^{2,k} & \text{if } m \notin Q \text{ and } m = jk, \text{ for } j \in \{1, ..., J\} \text{ and } k \in \{1, ..., K\} \end{cases}$$

Hence $M^3 \in \mathbb{M}$.

D) [*7 points*] Demonstrate that if the conditions you identified in part B) are not satisfied, $\mathbb{M}$ is not a vector space.

Suppose that $n = 1$ and $\mathbf{v}^0 = 1$. Let $Q = \{1\}$ and let $J = K = 2$. Consider the matrix $M = [1, 1, 1, 1]$ and note that $2M = [2, 2, 2, 2] \notin \mathbb{M}$.

E) [*4 points*] Assume now that your conditions guaranteeing that $\mathbb{M}$ is a vector space are satisfied. Write down the dimension of $\mathbb{M}$.

The dimension of $\mathbb{M}$ is nK.

F) [*7 points*] Continuing to assume that your conditions guaranteeing that $\mathbb{M}$ is a vector space are satisfied, write down a basis for $\mathbb{M}$.

For $i = 1, ..., n$, let $\mathbf{e}^i$ denote the i'th canonical vector, i.e., the vector whose i'th component is 1 and all other components are 0. Now for $i = 1, ..., n$ and $k = 1, ..., K$, let $M^{i,k}$ denote the matrix whose m'th column is $\begin{cases} 0 & \text{if } m \in Q \\ \mathbf{e}^i & \text{if } m \notin Q \text{ and } m = jk, \text{ for } j \in \{1, ..., J\} \\ 0 & \text{otherwise} \end{cases}$. The family of matrices $\{M^{i,k}\}_{\substack{i=1,...,n \\ k=1,...,K}}$ is a basis for $\mathbb{M}$.

Problem 3 [40 points]
Let

$$\begin{aligned} f(\mathbf{x}) &= x_1 + x_2, \\ g^1(\mathbf{x}) &= \max(0, x_1^3) - x_2, \\ g^2(\mathbf{x}) &= \max(0, x_1^3) + x_2 \end{aligned}$$

Now consider the following optimization problem

$$\max_{\mathbf{x} \in \mathbb{R}^2} \quad f(\mathbf{x}) \quad \text{s.t} \quad g^j(\mathbf{x}) \leq 0, \text{ for } j = 1, 2$$

Observe that all three functions are twice continuously differentiable. [It's not 100% obvious that g^j is differentiable when $x_1 = 0$; you can, however, simply accept this to be true. Knowing this, there's only one value that $\nabla g^j(0, x_2)$ can possibly take. Similarly, there is only one value that $Hg^j(0, x_2)$ can take. You can figure these out. (You can just assert them, don't have to prove that these values are correct.)]

A) [*7 points*] Using graphical methods *only*, indicate in a *clearly labeled* diagram
- for $j = 1, 2$, the lower contour set of g^j corresponding to zero
- the constraint set for this NPP
- the solution to this problem. (State its numeric value as well labeling.)
- a level set of the objective thru the solution.

Make sure that you clearly differentiate the constraint set from lower contour sets, using a different colored pen or some other device.

See Fig. 1.

B) [*3 points*] What property relating to second order conditions do the g^j's satisfy? You don't need to prove this property, you can just demonstrate it informally with a simple sketch. (Hint: this property has to do with all sets of a certain class being convex.)

They are both quasi-convex, i.e all lower contour sets are convex sets.

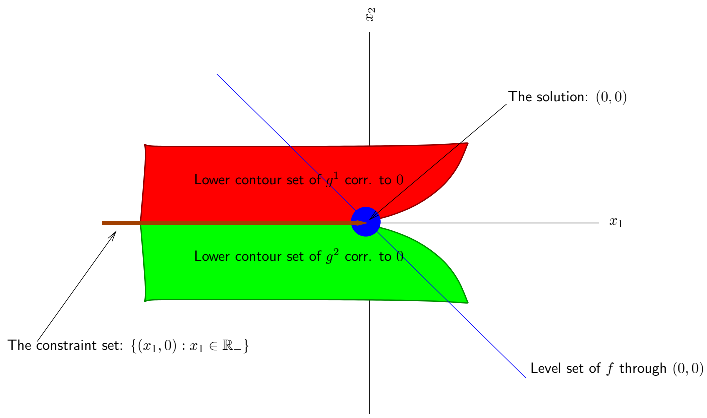

FIGURE 1. Solution to the NPP

C) [*4 points*] Write down a condition on the Hessian of g^j which is sufficient for the property you've identified in the previous part. (NB: The condition you write down should *not* include the words "determinant" or "bordered hessian".)

A sufficient condition for $f : \mathbb{R}^n \to \mathbb{R}$ to be quasi-convex is that for all $\mathbf{x}$ and all $\mathbf{dx}$ such that $\nabla f(\mathbf{x})'\mathbf{dx} = 0$, $\mathbf{dx}'\mathsf{Hf}(\mathbf{x})\mathbf{dx} \geq 0$.

D) [*13 points*] Set j equal to either 1 or 2, and verify that the condition you've written down in the previous part is satisfied by the Hessian of g^j.

$\nabla g^1(\mathbf{x}) = \left[3\max(0,x)^2, -1\right]$ and so $\nabla g^1(\mathbf{x}) \cdot \mathbf{dx} = 0$ iff for some $\alpha \in \mathbb{R}$, $\mathbf{dx} = \alpha\left[-1, 3\max(0,x)^2\right]$. In this case,

$$\begin{aligned}\mathbf{dx}'\mathsf{Hf}(\mathbf{x})\mathbf{dx} &= \alpha^2 \begin{bmatrix}-1 & 3\max(0,x)^2\end{bmatrix}\begin{bmatrix}6\max(x,0) & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}-1\\ 3\max(0,x)^2\end{bmatrix}\\ &= 6\alpha^2\max(x,0) \quad \geq \quad 0\end{aligned}$$

E) [*7 points*] At the solution to the problem, are the KT necessary conditions for a solution satisfied? Carefully explain your answer.

The KT necessary conditions are satisfied. These conditions state that if the CQ is satisfied, then .... In this instance, the CQ isn't satisfied, and unless the CQ is satisfied, the KT conditions impose no restrictions on the problem. To see that the CQ isn't satisfied, note that the linearized version of the constraint set is the entire horizontal line, $\{(x_1, 0) : x_1 \in \mathbb{R}\}$. The true constraint set is $\{(x_1, 0) : x_1 \in \mathbb{R}_-\}$. Therefore the linearized version of the problem does not accurately represent the true constraint.

F) [*7 points*] At the solution to the problem, are the KT sufficient conditions for a solution satisfied? Carefully explain your answer.

The sufficient conditions are not satisfied. The sufficient conditions state that if f is pseudo-concave and the g^j's are quasi-convex, then a sufficient condition for a solution to the Taylor is that the KT conditions are satisfied. In this problem the KT conditions are not satisfied. To see this, observe that at $\mathbf{x} = 0$, $\nabla g^1(\mathbf{x}) = [0, -1]$ and $\nabla g^2(\mathbf{x}) = [0, 1]$. The nonnegative cone generated by these vectors is $\{(x_1, 0) : x_1 \in \mathbb{R}\}$. However, $\nabla f(\mathbf{x}) = [1, 1]$ which does not belong to this cone.

G) [*7 BONUS points*] In class I told you that a certain conjecture was almost certainly false, since if it had been true, then it would be all over the textbooks. At the time I told you that I didn't have a counter example to this conjecture. State the conjecture for which the example in this question is a counter example.

In class, we discussed the possibility that a sufficient condition for the CQ to be satisfied was that the g^j's were quasi-convex. The only reason why I thought this might be true was that in the Varian example for which CQ failed, the constraints were *not* quasi-convex. The present example is a counter example to this sufficiency conjecture: we've verified that the constraints are both quasi-convex, but the CQ fails.

Problem 4 [40 points]

Let $f : \mathbb{R} \to \mathbb{R}$, be a twice continuously differentiable function. Suppose that $f(\cdot)$ attains a strict local maximum at $\bar{x}$ but that there exists $dx > 0$ such that $f(\bar{x} + dx) > f(\bar{x})$. Assume also that $f''(\bar{x}) < 0$. *Each part of this question builds on some or all of the previous parts.* A good strategy would be to read the entire question carefully, then do the diagram required in part D)(a), then complete the rest of the question.

A) [*10 points*] Prove by applying "global Taylor" (Taylor-Lagrange) to the derivative $f'(\cdot)$ that there exists $\hat{\alpha} > 0$ such that for $\alpha \in (0, \hat{\alpha})$, $f'(\bar{x} + \alpha dx) < 0$. (Hint: (1) this property would not necessarily hold if f were twice differentiable but not twice continuously differentiable; (2) you need to do a zero'th order Taylor expansion of f' about $\bar{x}$)

Since $f''(\bar{x}) < 0$ and $f''(\cdot)$ is continuous, there exists $\epsilon > 0$, such that for $x \in B(\bar{x}, \epsilon)$, $f''(x) < 0$. Pick $\hat{\alpha} > 0$ sufficiently small that $\bar{x} + \hat{\alpha} dx \in B(\bar{x}, \epsilon)$. Now pick $\alpha \in (0, \hat{\alpha})$ By the Taylor Lagrange theorem, $f'(\bar{x} + \alpha dx) = f'(\bar{x}) + f''(\bar{x} + \lambda \alpha dx)dx$, for some $\lambda \in [0, 1]$. Since $\lambda \leq 1$, $\bar{x} + \lambda \alpha dx \in B(\bar{x}, \epsilon)$, so that $f''(\bar{x} + \lambda \alpha dx) < 0$. Moreover, since f is maximized at $\bar{x}$, $f'(\bar{x}) = 0$. Finally, $dx > 0$ by assumption. Hence $f'(\bar{x}) + f''(\bar{x} + \lambda \alpha dx)dx < 0$.

B) [*10 points*] Let $\hat{\beta} = \inf\{\beta > 0 : f''(\bar{x} + \beta dx) \geq 0\}$. Use global Taylor on f to prove that for all $\beta \in (0, \hat{\beta}]$, $f(\bar{x} + \beta dx) < f(\bar{x})$. (Hint: given β, chose an appropriate $\epsilon \in (0, \beta)$, write $\bar{x} + \beta dx = (\bar{x} + \epsilon dx) + (\beta - \epsilon)dx$ and do a first order Taylor expansion of f around $f(\bar{x} + \epsilon dx)$.)

From the previous part, if $\epsilon > 0$ is sufficiently small, then $f'(\bar{x} + \epsilon dx) < 0$ and, since $f(\cdot)$ is locally maximized at $\bar{x}$, $f(\bar{x} + \epsilon dx) < f(\bar{x})$. Now fix $\beta \in (0, \hat{\beta})$.

$$f(\bar{x} + \beta dx) - f(\bar{x}) \quad < \quad f(\bar{x} + \beta dx) - f(\bar{x} + \epsilon dx)$$

which, for some $\lambda \in [0, 1]$

$$\begin{aligned} &= f'(\bar{x} + \epsilon dx) \; + \; 0.5 f''\big(\bar{x} + \epsilon dx + \lambda(\beta - \epsilon)dx\big)\big((\beta - \epsilon)dx\big)^2 \\ &= f'(\bar{x} + \epsilon dx) \; + \; 0.5 f''\big(\bar{x} + (\lambda\beta + (1 - \lambda)\epsilon dx)\big)\big((\beta - \epsilon)dx\big)^2 \\ &\leq 0 \end{aligned}$$

The last inequality holds because, from the previous part, we know that $f'(\bar{x} + \epsilon dx) < 0$ and because $\epsilon < \beta$ and $\lambda \in [0, 1]$, $\lambda\beta + (1 - \lambda)\epsilon \leq \beta \leq \hat{\beta}$, so that $f''(\bar{x} + (\lambda\beta + (1 - \lambda)\epsilon dx)) \leq 0$.

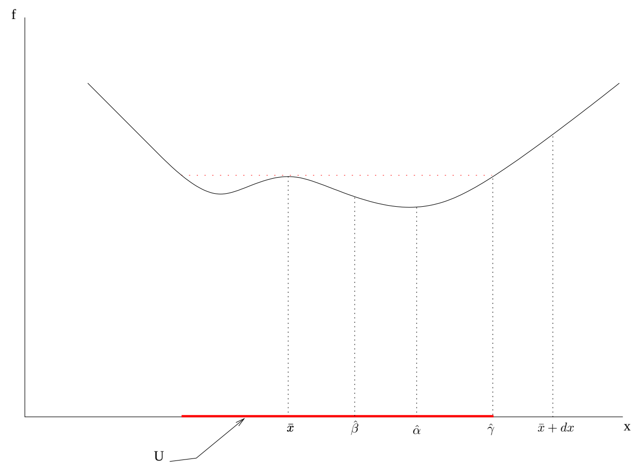

FIGURE 2. Graph of the Taylor question

C) [*10 points*] Let $\hat{\gamma} = \inf\{\gamma > 0 : f(\bar{x} + \gamma dx) > f(\bar{x})\}$. Write $f(\bar{x} + \hat{\gamma} dx)$ as a first order Taylor expansion of f about $\bar{x}$ plus a remainder term. Specify *explicitly* what the remainder term is, i.e., in this instance, you *know* which point in the domain to evaluate $f''(\cdot)$ to obtain the remainder term.

Because f is continouous, $f(\bar{x}+\hat{\gamma}dx) = 0$. Moreover, by the Taylor-Lagrange theorem, for some $\lambda \in [0, 1]$,

$$f(\bar{x} + \hat{\gamma}dx) \;-\; f(\bar{x}) \quad = \quad f'(\bar{x})\hat{\gamma}dx \;+\; 0.5f''(\bar{x} + \lambda\hat{\gamma}dx)(\hat{\gamma}dx)^2$$

Since the left hand size is zero, as is $f'(\bar{x})\hat{\gamma}dx$, it follows that $f''(\bar{x} + \lambda\hat{\gamma}dx) = 0$. From the previous part, we know that $f''(\bar{x} + \hat{\beta}dx) = 0$. Therefore, we have

$$f(\bar{x} + \hat{\gamma}dx) \;-\; f(\bar{x}) \quad = \quad f'(\bar{x})\hat{\gamma}dx \;+\; 0.5f''(\bar{x} + \hat{\beta}dx)(\hat{\gamma}dx)^2$$

D) [*10 points*] Let U denote the lower contour set of f corresponding to $f(\bar{x})$.

(a) Draw a graph of a function f that exhibits the properties described at the beginning of this question. Indicate on your graph the locations of $\bar{x}$, $\bar{x} + dx$, $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ and U.

See Fig. 2.

(b) Prove that there exists y in the interior of U such that $f''(y) \geq 0$.

By definition of $\hat{\gamma}$, U contains the open interval $(\bar{x}, \bar{x} + \hat{\gamma}dx)$. From Part B, we know that $f(\bar{x} + \hat{\beta}dx) < f(\bar{x})$, so that $\bar{x} + \hat{\beta}dx$ belongs to the interior of U. By definition of $\hat{\beta}$, $f''(\bar{x} + \hat{\beta}dx) \geq 0$.

Problem 5 [40 points]
Consider the following general equilibrium system. For $i = 1, 2$, supply of good i is given by $S^i(p_i, t_i)$, where p_i is the price of good i, and $t_i \geq 0$ is an ad valorum tax payable by producers. The consumer' demand function for good i is $D^i(p_i, y)$, where y denotes the consumer's income.

A) [*2 points*] Identify conditions on the primitives of this problem (i.e., on supply and demand) that are sufficient to ensure that the implicit function theorem can be applied. For the remainder of the question, you can assume that these conditions are satisfied.

For each good, the supply curve is upward sloping; the demand curve is downward sloping. (There are lots of other possible answers to this part.)

B) [*8 points*] Using the implicit function theorem, write down *in matrix form* an expression for the impact of changes in the exogenous variables (taxes and income) on equilibrium prices. Simplify your matrices to the extent that's possible with the information that you have at this point.

By the implicit function, we have

$$\begin{bmatrix} \frac{dp_1}{dt_1} & \frac{dp_1}{dt_2} & \frac{dp_1}{dy} \\ \frac{dp_2}{dt_1} & \frac{dp_2}{dt_2} & \frac{dp_2}{dy} \end{bmatrix} = -\begin{bmatrix} \frac{dS^1}{dp_1} - \frac{dD^1}{dp_1} & \frac{dS^1}{dp_2} - \frac{dD^1}{dp_2} \\ \frac{dS^2}{dp_1} - \frac{dD^2}{dp_1} & \frac{dS^2}{dp_2} - \frac{dD^2}{dp_2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{dS^1}{dt_1} & \frac{dS^1}{dt_2} & -\frac{dD^1}{dy} \\ \frac{dS^2}{dt_1} & \frac{dS^2}{dt_2} & -\frac{dD^2}{dy} \end{bmatrix}$$

$$= -\begin{bmatrix} \frac{dS^1}{dp_1} - \frac{dD^1}{dp_1} & 0 \\ 0 & \frac{dS^2}{dp_2} - \frac{dD^2}{dp_2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{dS^1}{dt_1} & 0 & -\frac{dD^1}{dy} \\ 0 & \frac{dS^2}{dt_2} & -\frac{dD^2}{dy} \end{bmatrix}$$

C) [*8 points*] Now, and for the remainder of the question, let $S^i(p_i, t_i) = \alpha_i p_i(1 - t_i)$, where $\alpha_i > 0$ is the slope of the supply function, and $D^i(p_i, y) = \frac{y}{\beta_i p_i}$. Using this specification, further simplify to the maximum extent possible the expression that you wrote down in part B) (i.e., compute the actual derivatives, and compute the inverse).

Writing $\Psi_i = \alpha_i(1 - t_i) + \frac{y}{\beta_i p_i^2}$, for $i = 1, 2$, we have

$$\begin{bmatrix} \frac{dp_1}{dt_1} & \frac{dp_1}{dt_2} & \frac{dp_1}{dy} \\ \frac{dp_2}{dt_1} & \frac{dp_2}{dt_2} & \frac{dp_2}{dy} \end{bmatrix} = \begin{bmatrix} \Psi_1 & 0 \\ 0 & \Psi_2 \end{bmatrix}^{-1} \begin{bmatrix} \alpha_1 p_1 & 0 & \frac{1}{\beta_1 p_1} \\ 0 & \alpha_2 p_2 & \frac{1}{\beta_2 p_2} \end{bmatrix}$$

$$= \begin{bmatrix} \Psi_1^{-1} & 0 \\ 0 & \Psi_2^{-1} \end{bmatrix} \begin{bmatrix} \alpha_1 p_1 & 0 & \frac{1}{\beta_1 p_1} \\ 0 & \alpha_2 p_2 & \frac{1}{\beta_2 p_2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\alpha_1 p_1}{\Psi_1} & 0 & (\beta_1 p_1 \Psi_1)^{-1} \\ 0 & \frac{\alpha_2 p_2}{\Psi_2} & (\beta_2 p_2 \Psi_2)^{-1} \end{bmatrix}$$

D) [*6 points*] Find the equilibrium of this system for the case in which

- there are no taxes
- for $i = 1, 2$, $\alpha_i = [\frac{4}{3}, 16]$,
- the lady's income is 100, and her $\boldsymbol{\beta} = [3/4, 1/4]$.

In this case the equilibrium value of $\mathbf{p}$, denoted $\mathbf{p}^*$ satisfies:

$$
\begin{aligned}
S^1(p_1^*, y) &= 4/3p_1 &= D^1(p_1^*, y) &= 400/3p_1^* \\
S^2(p_2^*, y) &= 16p_2 &= D^2(p_2^*, y) &= 400/p_2^*
\end{aligned}
$$

so that $\mathbf{p} = [5, 10]$.

E) [*8 points*] Now assume that taxes on suppliers are spent in part on enhancing National Public Radio, which distracts the consumer from working and hence lowers her income. The relationship between income and taxes is given by $y = y_0 - \sum_{i=1}^{2} \gamma_i t_i$, where $y_0 > 0$ and $\gamma_i > 0$. (This is an incredibly stupid economic model, but since ARE211 is a math class, this is legal.) Rewrite your answer to C, reducing the number of exogenous variables in the model by one (i.e., treat y_0 as a parameter, not an exogenous variable).

The new equilibrium condition becomes

$$
0 = \alpha_i p_i (1 - t_i) - \frac{y_0 - \sum_{i=1}^{2} \gamma_i t_i}{\beta_i p_i}
$$

so that

$$
\begin{aligned}
\begin{bmatrix} \frac{dp_1}{dt_1} & \frac{dp_1}{dt_2} \\ \frac{dp_2}{dt_1} & \frac{dp_2}{dt_2} \end{bmatrix}
&= - \begin{bmatrix} \Psi_1 & 0 \\ 0 & \Psi_2 \end{bmatrix}^{-1} \begin{bmatrix} \frac{\gamma_1}{\beta_1 p_1} - \alpha_1 p_1 & \frac{\gamma_2}{\beta_1 p_1} \\ \frac{\gamma_1}{\beta_2 p_2} & \frac{\gamma_2}{\beta_2 p_2} - \alpha_2 p_2 \end{bmatrix} \\
&= \begin{bmatrix} \Psi_1^{-1} & 0 \\ 0 & \Psi_2^{-1} \end{bmatrix} \begin{bmatrix} \alpha_1 p_1 - \frac{\gamma_1}{\beta_1 p_1} & -\frac{\gamma_2}{\beta_1 p_1} \\ -\frac{\gamma_1}{\beta_2 p_2} & \alpha_2 p_2 - \frac{\gamma_2}{\beta_2 p_2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\alpha_1 \beta_1 p_1^2 - \gamma_1}{\beta_1 p_1 \Psi_1} & -\frac{\gamma_2}{\beta_1 p_1 \Psi_1} \\ -\frac{\gamma_1}{\beta_2 p_2 \Psi_2} & \frac{\alpha_2 \beta_2 p_2^2 - \gamma_2}{\beta_2 p_2 \Psi_2} \end{bmatrix}
\end{aligned}
$$

F) [*8 points*] Use your answer to E) to obtain necessary and sufficient conditions for a sufficiently small equal increase in both taxes to result in an increase in the price of good 1 and a decrease in good 2.

Suppose that $dt_1 = dt_2 = d\tau$. In this case

$$\begin{bmatrix} dp_1 \\ dp_2 \end{bmatrix} = \begin{bmatrix} \left(\frac{dp_1}{dt_1} + \frac{dp_1}{dt_2}\right) d\tau \\ \left(\frac{dp_2}{dt_1} + \frac{dp_2}{dt_2}\right) d\tau \end{bmatrix}$$

$$= \begin{bmatrix} (\beta_1 p_1 \Psi_1)^{-1} \left(\alpha_1 \beta_1 p_1^2 - \sum_{i=1}^2 \gamma_i\right) \\ (\beta_2 p_2 \Psi_2)^{-1} \left(\alpha_2 \beta_2 p_2^2 - \sum_{i=1}^2 \gamma_i\right) \end{bmatrix} d\tau$$

so that

$$dp_1 \; > \; 0 \; > \; dp_2 \qquad \text{iff} \qquad \alpha_1 \beta_1 p_1^2 \; > \; \sum_{i=1}^{2} \gamma_i \; > \; \alpha_2 \beta_2 p_2^2$$