PROBLEM SET #10- ANSWER KEY

FIRST NPP PROBLEM SET

(1) Consider the following maximization problem (solve it graphically):

$\max_{x_1,x_2} f(x_1,x_2)$ with $f(x_1,x_2) = -x_1$

subject to $g_1 : -x_1^3 + x_2 \leq 0$ and $g_2 : -x_1^3 - x_2 \leq 0$.

a) What is the solution to the maximization problem?

Ans: The feasible set is displayed in the left graph of figure 1. Since it is the task to maximize $-x_1$, one has to pick the point with the lowest x_1 value in the feasible set, which is the point (0,0).

b) Are the KKT conditions satisfied for the solution to part a). If yes, write the gradient of the objective function as a postive linear combination of the gradients of the contstraints that are satisfied with equality. If not, explain why?

Ans: The gradient of the objective function and the gradient of the constraints that are satisfied with equality are displayed in the right side of figure 1. Note that the gradient of the objective function does *not* lie in the nonnegative cone defined by the gradients of the constraints that are satified with equality. The reason is that the constraint qualification is not satisfied, i.e., the two gradients of the constraints satified with equality are collinear.

c) Now, slightly change the problem and let the second constraint be $g_2 : -x_1^3 - \epsilon x_1 - x_2 \leq 0$ for $\epsilon > 0$ Again, what is the solution to your problem?

Ans: The feasible set is displayed in the upper graph of figure 1 for $\epsilon = 0.1$. Since it is the task to maximize $-x_1$, one has to pick the point with the lowest x_1 value in the feasible set, which is again the point (0,0).

d) For the revised problem in part c), is the Mantra satisfied. If yes, write the gradient of the objective function as a postive linear combination of the gradients of the contstraints that are satisfied with equality. If not, explain why?

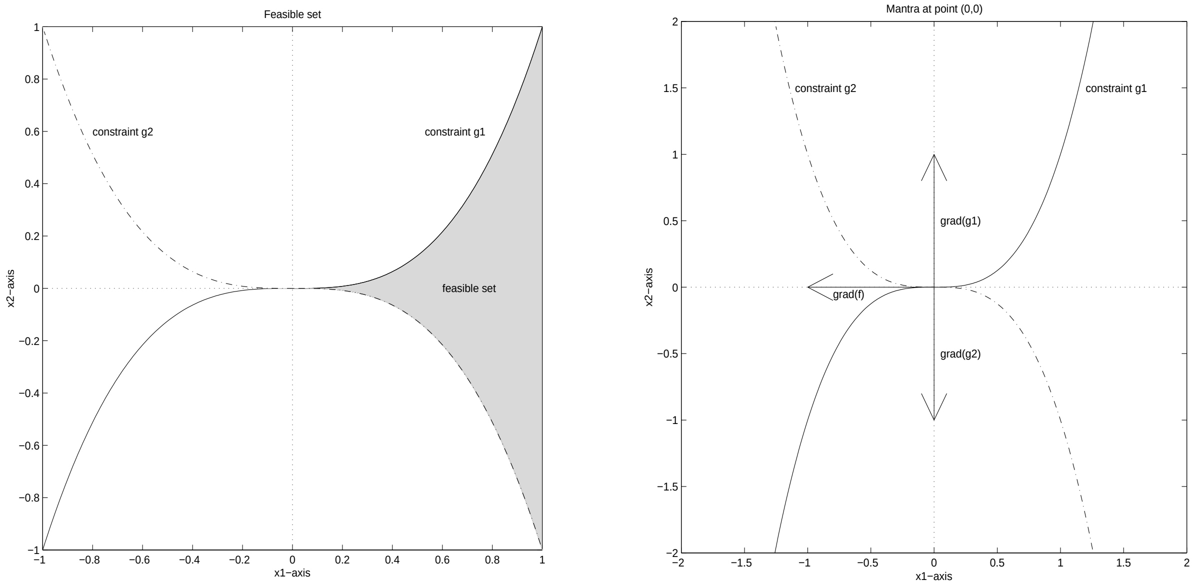

FIGURE 1. Feasible set (left) and Mantra at point (0,0) (right)

Ans: The gradient of the objective function and the gradient of the constraints that are satisfied with equality are displayed in the lower graph of figure 1. Now, the gradient of the objective function does lie in the nonnegative cone defined by the gradients of the constraints that are satified with equality.

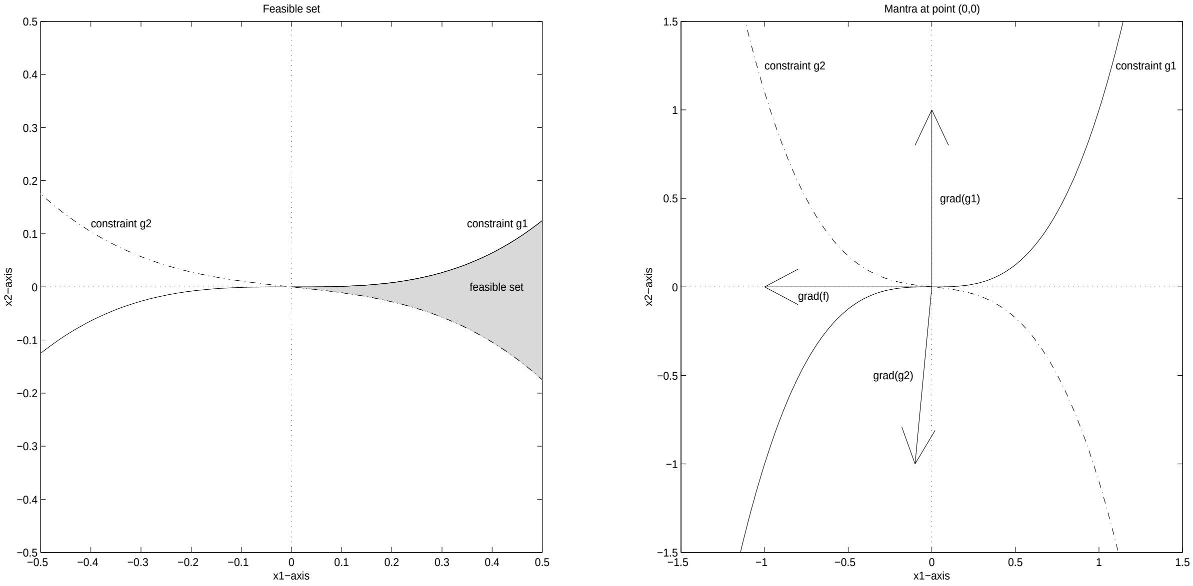

FIGURE 2. Feasible set (top) and Mantra at point (0,0) (bottom)

$$\nabla f = \frac{1}{\epsilon}\nabla g_1 + \frac{1}{\epsilon}\nabla g_2$$

(2) Consider the following minimization problem

$$\min_{x_1,x_2} 2x_1^2 + 2x_2^2 - 2x_1x_2 - 9x_2$$

subject to the following constraint set:

$$\begin{aligned} g_1 :&\quad x_1 &&\geq 0 \\ g_2 :&\quad x_2 &&\geq 0 \\ g_3 :&\quad 4x_1 + 3x_2 &&\leq 10 \\ g_4 :&\quad -4x_1^2 + x_2 &&\geq -2 \end{aligned}$$

Write down the Lagrangian and solve the first order necessary condition.

Hint: (1) show that the cases $x_1 = 0, x_2 > 0$ and $x_1 > 0, x_2 = 0$ and $x_1 = x_2 = 0$ lead to a contradiction. Infer that a possible solution must satisfy: $x_1 > 0, x_2 > 0$. **Note well:** You can only make this inference once you have checked in each of the three cases that the constraint qualification is satisfied. Otherwise you haven't excluded the possibility that you've reached a contradiction even though a solution does exist.

Ans: Consider the equivalent maximization problem in standard form:

$$\max_{x_1,x_2} -2x_1^2 - 2x_2^2 + 2x_1x_2 + 9x_2$$

subject to the following constraint set:

$$\begin{array}{lcl} g_1: & -x_1 & \leq 0 \\ g_2: & -x_2 & \leq 0 \\ g_3: & 4x_1+3x_2 & \leq 10 \\ g_4: & 4x_1^2-x_2 & \leq 2 \end{array}$$

Note: The gradient of the objective function is: $\nabla f = \begin{pmatrix} -4x_1+2x_2 \\ -4x_2+9 \end{pmatrix}$.

Hence the gradient vanishes at the point $(\frac{9}{8}, \frac{9}{4})$ which is oustide the feasible set as $g_3(\frac{9}{8}, \frac{9}{4}) = 11.25 > 10$.

Since the gradient is non-vanishing in the feasible set the maximum must occur on the border, i.e., where at least one constraint is satified with equality.

For future reference, the Jacobian of the constraints is: $\begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 4 & 3 \\ 4x_1 & -1 \end{bmatrix}$

Consequently the Lagrangian becomes:

$$L = -2x_1^2 - 2x_2^2 + 2x_1x_2 + 9x_2 + \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3(10 - 4x_1 - 3x_2) + \lambda_4(2 - 4x_1^2 + x_2)$$

The first order necessary conditions are:

$$\begin{array}{lll} (1)\ \frac{\delta L}{\delta x_1} = -4x_1 + 2x_2 + \lambda_1 - 4\lambda_3 - 8\lambda_4 x_1 = 0 & & \\ (2)\ \frac{\delta L}{\delta x_2} = -4x_2 + 2x_1 + 9 + \lambda_2 - 3\lambda_3 + \lambda_4 = 0 & & \\ (3)\ \frac{\delta L}{\delta \lambda_1} = x_1 \geq 0 & (7)\ \lambda_1 x_1 = 0 & (11)\ \lambda_1 \geq 0 \\ (4)\ \frac{\delta L}{\delta \lambda_2} = x_2 \geq 0 & (8)\ \lambda_2 x_2 = 0 & (12)\ \lambda_2 \geq 0 \\ (5)\ \frac{\delta L}{\delta \lambda_3} = 10 - 4x_1 - 3x_2 \geq 0 & (9)\ \lambda_3(10 - 4x_1 - 3x_2) = 0 & (13)\ \lambda_3 \geq 0 \\ (6)\ \frac{\delta L}{\delta \lambda_4} = 2 - 4x_1^2 + x_2 \geq 0 & (10)\ \lambda_4(2 - 4x_1^2 + x_2) = 0 & (14)\ \lambda_4 \geq 0 \end{array}$$

(2) Look at the 4 positivity cases for λ_3, λ_4 (i.e., where each of them is strictly greater than zero or zero).

Ans:

a) First, consider the 4 positivity cases for x_1, x_2

case I: Assume $x_1 = 0, x_2 > 0$.

- (i) From (8) and (10) we therefore know that $\lambda_2 = \lambda_4 = 0$.
- (ii) If we assume $\lambda_3 = 0$ we know from (1) that $\lambda_1 = -2x_2 < 0$ which contradicts (11). Hence $\lambda_3 > 0$.
- (iii) Using $\lambda_3 > 0$ and $x_1 = 0$ in (9) yields $x_2 = \frac{10}{3}$.
- (iii) However, using $x_1 = 0$ and (i) and (iii) in (2) implies that $\lambda_3 = \frac{1}{3}(9 - 4 * \frac{10}{3}) = -\frac{13}{9} < 0$ which contradicts (13).

case II: Assume $x_1 > 0, x_2 = 0$.

- (i) From (7) we therefore know that $\lambda_1 = 0$.
- (ii) Using $x_2 = 0$ as well as (i), (13), and (14) in (1) we therefore know: $-4x_1 + 2x_2 + \lambda_1 - 4\lambda_3 - 8\lambda_4 x_1 \leq -4x_1 < 0$ which gives us again a contradiction.

Check the constraint qualification: if $x_2 = 0$, then the two constraints that are satisfied with equality are g_2 and g_4 (since g_4 is satisfied with equality when $x_2 = \sqrt{1/2}$, at which point g_3 is satisfied with strict inequality). $x_2 = 10/3$. The matrix defined by the gradients of these two constraints is $\begin{bmatrix} 0 & -1 \\ 4\sqrt{1/2} & -1 \end{bmatrix}$ which has full rank, so the CQ is satisfied.

case III: Assume $x_1 = x_2 = 0$.

- (i) From (9) and (10) we therefore know that $\lambda_3 = \lambda_4 = 0$.
- (ii) However, using $x_1 = x_2 = 0$ and (i) in (2) yields $\lambda_2 = -9$ which contradicts (12).

Check the constraint qualification: if the $\mathbf{x}$ vector is strictly positive, then the two constraints that are satisfied with equality are g_3 and g_4 The matrix defined by the gradients of these two constraints is $\begin{bmatrix} 4 & 3 \\ 4x_1 & -1 \end{bmatrix}$ which, since $x_1 > 0$, has full rank so the CQ is satisfied.

Hence the only possible solution is: $x_1 > 0, x_2 > 0$. (15)

From (7) and (8) we know that $\underline{\underline{\lambda_1 = \lambda_2 = 0}}$ (16)

b) Second, consider the 4 positivity cases for λ_3, λ_4

case I: Assume $\lambda_3 > 0, \lambda_4 > 0$.

- (i) From (9) we know $10 - 4x_1 - 3x_2 = 0$ and from (10) we know that $2 - 4x_1^2 + x_2 = 0$.
- (ii) Adding three times the second equation in (i) to the first yields $16 - 12x_1^2 - 4x_1 = 0 \Leftrightarrow 12(x_1 - 1)(x_1 + \frac{4}{3}) = 0$. Since $x_1 > 0$ after (15) the only possible solution is $x_1 = 1 \Rightarrow x_2 = 2$.
- (iii) Using (ii) and (16) in (1) yields $-4(\lambda_3 + 2\lambda_4) = 0$ which contradicts the inital assumption of $\lambda_3 > 0, \lambda_4 > 0$

case II: Assume $\lambda_3 = 0, \lambda_4 > 0$.

- (i) From (10) we know that $2 - 4x_1^2 + x_2 = 0 \Leftrightarrow x_2 = 4x_1^2 - 2$
- (ii) Using $\lambda_3 = 0$ as well as (16) and (i) in (2) we get $-4(4x_1^2 - 2) + 2x_1 + 9 + \lambda_4 = 0 \Leftrightarrow -16x_1^2 + 2x_1 + 17 + \lambda_4 = 0$.
  The quadratic equation has one negative solution (that violates (3)) and $x_1 = \frac{-1}{16} + \frac{\sqrt{4+4*16(17+\lambda_4)}}{16} \geq \frac{-1}{16} + \frac{\sqrt{4+4*16*17}}{16} \geq 2$
- (iii) Using (ii) in (i) we know that $x_2 \geq 14$.
- (iv) However $x_1 \geq 2, x_2 \geq 14$ clearly violate (5).

case III: Assume $\lambda_3 = \lambda_4 = 0$.

- (i) Using $\lambda_3 = \lambda_4 = 0$ and (16) in (1) we obtain $-4x_1 + 2x_2 = 0$ and in (2) we get $-4x_2 + 2x_1 + 9 = 0$.
- (ii) Adding twice the the second equation in (i) to the first we get $-6x_2 + 18 = 0$ or $x_2 = 3$. Hence from (i) $x_1 = \frac{3}{2}$.
- (iii) However $x_1 = \frac{3}{2}, x_2 = 3$ contradicts (5).

Hence the only possible solution is: $\lambda_3 > 0$ $\underline{\underline{\lambda_4 = 0}}$. (17)

From (9) we know $10 - 4x_1 - 3x_2 = 0 \Leftrightarrow x_1 = \frac{5}{2} - \frac{3}{4}x_2$ (18)

Using (18) in (1) we get: $5x_2 = 10 + 4\lambda_4$ (19)

Using (18) in (2) we get: $\frac{11}{2}x_2 = 14 - 3\lambda_4$ (20)

Add three times (19) to four times (20): $15x_2 + 22x_2 = 86 \Leftrightarrow \underline{\underline{x_2 = \frac{86}{37}}}$

Hence from (18) $x_1 = \frac{5}{2} - \frac{3}{4} * \frac{86}{37} = \frac{185-129}{74} = \frac{56}{74}$ and $\underline{\underline{x_1 = \frac{28}{37}}}$

And from (19) ($\lambda_3 = \frac{5}{4} * \frac{86}{37} - \frac{5}{2} = \frac{215-185}{74} = \frac{30}{74}$ and $\underline{\underline{\lambda_3 = \frac{15}{37}}}$

Note: Since the feasible set is compact and the objective function is continous, it must obtain a maximum. And there is only one potential point that satifies the necessary conditions. Hence it must be the unique maximum.

(3) Find the maximum and minimum distance from the origin to the ellipse $x_1^2 + x_1x_2 + x_2^2 = 3$.

(Hint: instead of using the distance $\sqrt{x_1^2 + x_2^2}$, maximize or minimize the square of the distance which is much easier)

a) Set up the Lagrangian and derive the first order necessary conditions.

Ans:

Note: The gradient of the objective function is: $\nabla f = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}$.
Hence the gradient vanishes at the point $(0,0)$ which is oustide the feasible set. Since the gradient is non-vanishing in the feasible set the Lagrange multiplier of the single equality constraint has to be non-negative: $\lambda \neq 0$ (hence we can divide by it) (1)

Consequently the Lagrangian becomes:

$$L = x_1^2 + x_2^2 + \lambda(3 - x_1^2 - x_1x_2 - x_2^2)$$

The first order necessary conditions are:

(2) $\frac{\delta L}{\delta x_1} = 2x_1 - 2\lambda x_1 - \lambda x_2 = 0 \;\Leftrightarrow\; 2x_1 = \lambda(2x_1 + x_2)$

(3) $\frac{\delta L}{\delta x_2} = 2x_2 - 2\lambda x_2 - \lambda x_1 = 0 \;\Leftrightarrow\; 2x_2 = \lambda(2x_2 + x_1)$

(4) $\frac{\delta L}{\delta \lambda} = 3 - x_1^2 - x_1x_2 - x_2^2 = 0$

b) Solve the first order necessary conditions.

Ans: From (2) we know that $\lambda = \frac{2x_1}{2x_1+x_2}$ (5)

Dividing (2) by (3) yields: $\frac{x_1}{x_2} = \frac{2x_1+x_2}{2x_2+x_1}$

Cross-multiplying gives: $2x_1x_2 + x_1^2 = 2x_1x_2 + x_2^2 \;\Leftrightarrow\; x_1^2 = x_2^2$

Let's denote the distance to the origin by $M = \sqrt{x_1^2 + x_2^2}$

case I: $x_1 = x_2$

Using $x_2 = x_1$ in (4) results in: $3 = 3x_1^2$.

Hence $\underline{\underline{x_1^{(1)} = 1,\ x_2^{(1)} = 1,\ \lambda^{(1)} = \frac{2}{3},\ M = \sqrt{2} \approx 1.4142}}$

and $\underline{\underline{x_1^{(2)} = -1,\ x_2^{(2)} = -1,\ \lambda^{(2)} = \frac{2}{3},\ M = \sqrt{2} \approx 1.4142}}$

case II: $x_1 = -x_2$

Using $x_2 = -x_1$ in (4) results in: $3 = x_1^2$.

Hence $\underline{\underline{x_1^{(3)} = \sqrt{3},\ x_2^{(3)} = -\sqrt{3},\ \lambda^{(3)} = 2,\ M = \sqrt{6} \approx 2.4495}}$

and $\underline{\underline{x_1^{(4)} = -\sqrt{3},\ x_2^{(4)} = \sqrt{3},\ \lambda^{(4)} = 2,\ M = \sqrt{6} \approx 2.4495}}$

c) Check the bordered Hessian for the second-order sufficiency conditions.

Ans: From (2) we know that $\lambda = \frac{2x_1}{2x_1+x_2}$ (5)

Dividing (2) by (3) yields: $\frac{x_1}{x_2} = \frac{2x_1+x_2}{2x_2+x_1}$

Cross-multiplying gives: $2x_1x_2 + x_1^2 = 2x_1x_2 + x_2^2 \;\Leftrightarrow\; x_1^2 = x_2^2$

Let's denote the distance to the origin by $M = \sqrt{x_1^2 + x_2^2}$

case I: $x_1 = x_2$

Using $x_2 = x_1$ in (4) results in: $3 = 3x_1^2$.

Hence $\underline{\underline{x_1^{(1)} = 1,\ x_2^{(1)} = 1,\ \lambda^{(1)} = \frac{2}{3},\ M = \sqrt{2} \approx 1.4142}}$

and $\underline{\underline{x_1^{(2)} = -1,\ x_2^{(2)} = -1,\ \lambda^{(2)} = \frac{2}{3},\ M = \sqrt{2} \approx 1.4142}}$

case II: $x_1 = -x_2$

Using $x_2 = -x_1$ in (4) results in: $3 = x_1^2$.

Hence $\underline{\underline{x_1^{(3)} = \sqrt{3},\ x_2^{(3)} = -\sqrt{3},\ \lambda^{(3)} = 2,\ M = \sqrt{6} \approx 2.4495}}$

and $\underline{\underline{x_1^{(4)} = -\sqrt{3},\ x_2^{(4)} = \sqrt{3},\ \lambda^{(4)} = 2,\ M = \sqrt{6} \approx 2.4495}}$

c) The bordered Hessian becomes: $\begin{pmatrix} 0 & 2x_1+x_2 & 2x_2+x_1 \\ 2x_1+x_2 & 2-2\lambda & -\lambda \\ 2x_2+x_1 & -\lambda & 2-2\lambda \end{pmatrix}$

Let's plug in the different values for $(x_1^{(i)}, x_2^{(i)}, \lambda^{(i)})$:

(1) $x_1^{(1)} = 1,\ x_2^{(1)} = 1,\ \lambda^{(1)} = \frac{2}{3}$:
The second order leading principal minor is:
$\begin{vmatrix} 0 & 3 & 3 \\ 3 & \frac{2}{3} & -\frac{2}{3} \\ 3 & -\frac{2}{3} & \frac{2}{3} \end{vmatrix}$ =-12 - 12 = -24 (develop after 1st column)
Hence the second order condition for a constraint min is fullfiled.

(2) $x_1^{(2)} = -1,\ x_2^{(2)} = -1,\ \lambda^{(2)} = \frac{2}{3}$:
The second order leading principal minor is:
$\begin{vmatrix} 0 & -3 & -3 \\ -3 & \frac{2}{3} & -\frac{2}{3} \\ -3 & -\frac{2}{3} & \frac{2}{3} \end{vmatrix}$ =-12 - 12 = -24 (develop after 1st column)
Hence the second order condition for a constraint min is fullfiled.

(3) $x_1^{(3)} = \sqrt{3},\ x_2^{(3)} = -\sqrt{3},\ \lambda^{(3)} = 2$:
The second order leading principal minor is:
$\begin{vmatrix} 0 & \sqrt{3} & -\sqrt{3} \\ \sqrt{3} & -2 & -2 \\ -\sqrt{3} & -2 & -2 \end{vmatrix}$ =12 + 12 = 24 (develop after 1st column)
Hence the second order condition for a constraint max is fullfiled.

(4) $x_1^{(4)} = -\sqrt{3},\ x_2^{(4)} = \sqrt{3},\ \lambda^{(4)} = 2$:
The second order leading principal minor is:
$\begin{vmatrix} 0 & -\sqrt{3} & \sqrt{3} \\ -\sqrt{3} & -2 & -2 \\ \sqrt{3} & -2 & -2 \end{vmatrix}$ =12 + 12 = 24 (develop after 1st column)
Hence the second order condition for a constraint max is fullfiled.