

PROBLEM SET #06

THIRD LIN ALGEBRA PROBLEM SET

Problem 1

$$\text{Let } \mathbf{A} = \begin{pmatrix} 3 & 3 & -1 & 0 \\ -2 & -3 & 1 & 0 \\ -6 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -6 \end{pmatrix}$$

a) Show that \mathbf{A} and \mathbf{B} are nonsingular.

b) Calculate \mathbf{A}^{-1} , \mathbf{B}^{-1} , $(\mathbf{AB})^{-1}$, and $(\mathbf{A}^T)^{-1}$.

c) Solve the linear equation system $\mathbf{Ax} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$ and $(\mathbf{AB})\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

Problem 2

Cramer's rule: Simon & Blume question 9.13 (page 196).

Problem 3

Show that if the matrix \mathbf{A} is nonsingular and symmetric, then the matrix \mathbf{A}^{-1} is also symmetric. (You can use as a fact that the left- and right inverse of the matrix \mathbf{A} are the same and that the inverse is unique).

Problem 4

Given an example of a 2x2 matrix \mathbf{A} such that the function $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$

- a) maps the unit circle to itself.
- b) maps the unit circle to a line in \mathbb{R}^2 .
- c) maps the unit circle to a single point.
- d) over several iterations, maps the unit circle to an ellipse the size of Miami.
- e) rotates every vector by 45 degrees (counter-clockwise) and stretches it by a factor of 2.

For each of the above, provide a sketch (or series of sketches) and an explanation.

Problem 5

Consider the function $f(x) = 20x^3 - 120x^2 - 5x + 36$

- What is the derivative of this function?
- What is the derivative of this function evaluated at $x_0 = 4$?
- What is the differential of this function at $x_0 = 4$?
- Approximate the change in the function when moving from $x_0 = 4$ to $x_1 = 5$.
- What is the actual change in the function when moving from $x_0 = 4$ to $x_1 = 5$?

Problem 6

Compute the directional derivative of the function $f(x, y) = xy^2 + x^3y$ at the point $(4, -2)$ in the direction $(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}})$.

Problem 7

Consider the function $f(x) = e^x$.

- Calculate the (i) third order and (ii) fourth order Taylor approximation of $f(\cdot)$ around the point $x = 0$.
- Approximate the value of $f(x)$ using the two Taylor approximations of part (a) for $x = 0.2$ and $x = 1$.
- Calculate the actual value of $f(x)$ for $x = 0.2$ and $x = 1$.

Problem 8

Recall the slightly different version of Taylor's theorem from Section:

If the function f is $(n+1)$ times continuously differentiable on $I = (a, b)$ then we know that for $x_0 \in I$ and for $x = x_0 + h \in I$:

$$f(x) = f(x_0 + h) = f(x_0) + \sum_{k=1}^n \frac{f^{(k)}(x_0)(x - x_0)^k}{k!} + R_n(x)$$

where $R_n(x) = f^{(n+1)}(\eta) \frac{(x-x_0)^{n+1}}{(n+1)!}$ for some η between x_0 and x .

- a) Show that a sufficient condition for f to attain a strict (local) maximum on I at x_0 is that the derivatives $f^{(k)}(x_0)$ are zero for $k = 1 \dots n$, and $f^{(n+1)}(x_0)$ is negative for some odd n .
- b) If $f^{(k)}(x_0)$ is zero for $k = 1 \dots n-1$ and $f^{(n)}(x_0)$ is non-zero, show that there exists an ϵ -neighborhood around x_0 where the absolute value of the n^{th} -order Taylor expansion is bigger than the absolute value of the remainder term $R_n(x)$.
- c) Show that the n^{th} -order Taylor approximation around any point x_0 of a polynomial of degree n (i.e. a function of the form $f(x) = \sum_{k=0}^n \alpha_k x^k$) is the function itself.

Problem 9

Consider the function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ (where \mathbf{A} is a $(n \times n)$ matrix and \mathbf{b} is a $(n \times 1)$ column vector).

Show that f is bijective $\Leftrightarrow \mathbf{A}$ is nonsingular.