Fall 2015 **ARE211**

Problem Set #02
Second Lin Algebra Problem Set
Due date: Sep 16

(1) Eigenvalues and definiteness:
   a) Let A be a symmetric (nxn) matrix with one or more negative eigenvalues. What can you say about the determinant of the matrix and its rank?
   b) Let A be symmetric (2x2) matrix. Is it true that A is indefinite iff its determinant is negative. Explain your answer (No formal proof necessary).

(2) Cramer's rule: Simon & Blume question 9.13 (page 196).

(3) Show that if the matrix $\mathbf{A}$ is nonsingular and symmetric, then the matrix $\mathbf{A}^{-1}$ is also symmetric. (You can use as a fact that the left- and right inverse of the matrix $\mathbf{A}$ are the same and that the inverse is unique).

(4) Given an example of a 2x2 matrix $\mathbf{A}$ such that the function $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$
   a) maps the unit circle to itself.
   b) maps the unit circle to a line in $\mathbb{R}^2$.
   c) maps the unit circle to a single point.
   d) over several iterations, maps the unit circle to an ellipse the size of Miami.
   e) rotates every vector by 45 degrees (counter-clockwise) and stretches it by a factor of 2.

   For each of the above, provide a sketch (or series of sketches) and an explanation.

(5) Let (M^n) be a sequence of 2×2 symmetric matrices. Assume that
   (i) there exists $\alpha \in (-1, 0)$ such that for all n, $\det(M^n) = \alpha$.
   (ii) there exists unit length vectors $\mathbf{v}^1, \mathbf{v}^2 \in \mathbb{R}^2$ s.t. for all n, $\mathbf{v}^1, \mathbf{v}^2$ are eigen-vectors for M^n.
   (iii) For each n, let $\lambda_1^n > 0$ be the eigenvalue corresponding to $\mathbf{v}^1$ for the matrix M^n. Assume that (λ_1^n) is a strictly decreasing sequence with $\lim_n \lambda_1^n = 0$.
   (iv) For each n, denote by λ_2^n the eigenvalue corresponding to $\mathbf{v}^2$ for the matrix M^n. Assume that $|\lambda_2^1| = |\lambda_1^1|$, i.e., the eigenvalues have the same absolute value for M^1.

   Questions:
   (a) Sketch the image of the unit circle $\mathbb{C}$ under M^n for three values of n including $n = 1$, say, for example $n = 1, 2, 4$. Your graph should reflect what you know about α. Include a sketch of the unit circle on your figure, for reference. Also, indicate the image of the eigenvectors on your graph. Label everything on your graph clearly so there's no ambiguity about what is what.
   (b) What can you say about the definiteness of each M^n?
   (c) Does (λ_2^n) contain a convergent subsequence? Justify your answer.
   (d) Let $P_+^n = \{\mathbf{x} \in \mathbb{C} : \mathbf{x}' M^n \mathbf{x} > 0\}$ and $P_-^n = \{\mathbf{x} \in \mathbb{C} : \mathbf{x}' M^n \mathbf{x} < 0\}$. What is:
   (i) $\cup_{n=1}^{\infty} P_+^n$? (ii) $\cap_{n=1}^{\infty} P_+^n$? (iii) $\cup_{n=1}^{\infty} P_-^n$? (iv) $\cap_{n=1}^{\infty} P_-^n$?
   You may prefer to draw a picture to illustrate some or all of these.

(6) Except for part (a), this question is harder than the previous ones.

(a) Given $N \in \mathbb{N}$, $N > 2$, we say that a nonempty set W is an N-wector space if $\{\mathbf{v}^1, ..., \mathbf{v}^N\} \subset W$ and $\alpha \in \mathbb{R}^N$ implies $\sum_{i=1}^N \alpha_i \mathbf{v}^i \in W$. Show that for any $N \in \mathbb{N}$, a set W is an N-wector space iff it is a vector space.

(b) The remaining parts of this question relate to the following construction. Fix $\boldsymbol{\theta} \in \mathbb{R}^5$ and a set $K \subset \mathbb{N}$. Let

$$X(\boldsymbol{\theta}, K) = \left\{ \text{sequences in } \mathbb{R}^5 \text{ s.t. } \begin{cases} x_n = \boldsymbol{\theta} & \text{for all } n \in K \\ x_{n,2} = x_{n,3} & \text{for all } n \in K^C \end{cases} \right\}$$

where $x_{n,j}$ denotes the $j'th$ component of the n'th element of the sequence. What is the *largest* collection of $\boldsymbol{\theta}$'s in $\mathbb{R}^5$ and largest collection of sets K's for which $X(\boldsymbol{\theta}, K)$ is a finite dimensional vector space. To get full marks for this question, you must prove that for the pair of collections that you have identified,

(i) whenever $(\boldsymbol{\theta}, K)$ belongs to this pair of collections, then $X(\boldsymbol{\theta}, K)$ is a finite dimensional vector space,

(ii) whenever $(\boldsymbol{\theta}, K)$ does not belong to this pair of collections, then $X(\boldsymbol{\theta}, K)$ is not a finite dimensional vector space,

(c) Fix a set $K \subset \mathbb{N}$ and $\boldsymbol{\theta} \in \mathbb{R}^5$ such that $X(\boldsymbol{\theta}, K)$ is a finite-dimensional vector space. Find a basis for $X(\boldsymbol{\theta}, K)$. Do this abstractly, not for a specific K and $\boldsymbol{\theta}$. That is, you should give one answer that "works" for all K and all $\boldsymbol{\theta}$ such that $X(\boldsymbol{\theta}, K)$ is a vector space. Demonstrate that it is a basis. Hint: it is quite possible that you have already partially or fully completed part c) in your answer to part b). If you have, simply refer to your previous answer; don't repeat work you've already done.

(d) Given a set $K \subset \mathbb{N}$ and $\boldsymbol{\theta} \in \mathbb{R}^5$ such that $X(\boldsymbol{\theta}, K)$ is a finite dimensional vector space, what is the dimension of $X(\boldsymbol{\theta}, K)$?

(e) Given a set $K \subset \mathbb{N}$ and $\boldsymbol{\theta} \in \mathbb{R}^5$ such that $X(\boldsymbol{\theta}, K)$ is a finite dimensional vector space, find a minimal spanning set for $X(\boldsymbol{\theta}, K)$ *that is not a basis.* Again, do this abstractly. Demonstrate that it spans, is minimal, but that it isn't a basis.

(7) Fix $n \in \mathbb{N}$ and a vector $\mathbf{v}^0 \in \mathbb{R}^n$, two natural numbers $J > 1$ and $K > 1$, and a nonempty set $Q \subset \{1, ...JK\}$. Now let $\mathbb{M}$ denote the set of all $n \times JK$ matrices M such that $M = [\mathbf{x}^1, ..., \mathbf{x}^{JK}]$ where

$$\mathbf{x}^m = \begin{cases} \mathbf{v}^0 & \text{if } m \in Q \\ \mathbf{v}^k & \text{if } m \notin Q \text{ and } m = k + jK, \text{ for } j \in \{0, ..., J-1\} \text{ and } k \in \{1, ..., K\} \end{cases}$$

for some $n \times K$ matrix $V = [\mathbf{v}^1, ..., \mathbf{v}^K]$. (Note that each distinct element of $\mathbb{M}$ is defined by a *different* matrix V.)

(a) Think of an example of an element of $\mathbb{M}$, for $K = 3$, $J = 4$, a set Q that has at least 3 elements and a $n \times K$ matrix V. (Keep it simple!!) Write down Q, n, $\mathbf{v}^0$ and V. Then write down your matrix M.

(b) Identify conditions under which $\mathbb{M}$ is a vector space.

(c) Demonstrate that if the conditions you identified in part 7b are satisfied, $\mathbb{M}$ is indeed a vector space.

(d) Demonstrate that if the conditions you identified in part 7b are not satisfied, $\mathbb{M}$ is not a vector space.

(e) Assume now that your conditions guaranteeing that $\mathbb{M}$ is a vector space are satisfied. Write down the dimension of $\mathbb{M}$.

(f) Continuing to assume that your conditions guaranteeing that $\mathbb{M}$ is a vector space are satisfied, write down a basis for $\mathbb{M}$.