Fall 2008 **ARE211**

PROBLEM SET #05- ANSWER KEY

SECOND LIN ALGEBRA PROBLEM SET

Problem 1

What is the difference between a minimum spanning set for a vector space and a basis for a vector space. Provide an example highlighting this difference.

Ans: The minimum spanning set does *not* have to be part of the vector space V, while a basis has to be an element of the vector set. For example, consider the vector space

$$V = \{\mathbf{x}|\mathbf{x} = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \alpha \in \mathbb{R}\}$$

which is a straight line through the origin.

A minimal spanning set for V are the two vectors:

$$\mathbf{v}^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \mathbf{v}^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Take away any of the two vectors and the remaining vector does not span V. However they are *not* a basis as neither $\mathbf{v}^1$ nor $\mathbf{v}^2$ are an element of V.

Problem 2

Simon & Blume question 11.2 (page 243). Explain your answer.

Ans: There are several ways to test for linear dependence of the vectors $\mathbf{v}^1$, $\mathbf{v}^2$, ..., $\mathbf{v}^n$. The definition of linear independence was

> Show that the linear equation system $\left[\mathbf{v}^1\ \mathbf{v}^2\ \cdots\ \mathbf{v}^n\right]\mathbf{x} = \mathbf{0}$ has the *only* soulution $\mathbf{x} = 0$. (Where $\mathbf{x}$ is a (nx1) column vector).

a) (i) Given: $\mathbf{v}^1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{v}^2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

(ii) Let's consider the linear equation system $\left[\mathbf{v}^1\ \mathbf{v}^2\right]\mathbf{x} = \mathbf{0}$

$$\begin{pmatrix} 2 & 1 & \vdots & 0 \\ 1 & 2 & \vdots & 0 \end{pmatrix}$$

Add (-2) times the second row to the first row:

$$\Leftrightarrow \begin{pmatrix} 0 & -3 & \vdots & 0 \\ 1 & 2 & \vdots & 0 \end{pmatrix}$$

Divide the first row by (-3):

$$\Leftrightarrow \begin{pmatrix} 0 & 1 & \vdots & 0 \\ 1 & 2 & \vdots & 0 \end{pmatrix}$$

Add (-2) times the first row to the second:

$$\Leftrightarrow \begin{pmatrix} 0 & 1 & \vdots & 0 \\ 1 & 0 & \vdots & 0 \end{pmatrix}$$

Hence we know from the first row that $x_1 = 0$ and from the second row that $x_2 = 0$

(iii) Since $\mathbf{0}$ is the only solution to the linear equation syatem in (ii), we know that $\mathbf{v}^1$ and $\mathbf{v}^2$ are linearly independent.

b) (i) Given: $\mathbf{v}^1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{v}^2 = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$

(ii) Let's consider the linear equation system $\left[\mathbf{v}^1\mathbf{v}^2\right]\mathbf{x} = \mathbf{0}$

$$\begin{pmatrix} 2 & -4 & \vdots & 0 \\ 1 & -2 & \vdots & 0 \end{pmatrix}$$

Add (-2) times the second row to the first row:

$$\Leftrightarrow \begin{pmatrix} 0 & 0 & \vdots & 0 \\ 1 & -2 & \vdots & 0 \end{pmatrix}$$

Clearly, $x_1 = 2$ and $x_2 = 1$ is a nonzero solution to the system

(iii) Since the linear equation system in (ii) has a nonzero solution, we know that $\mathbf{v}^1$ and $\mathbf{v}^2$ are linearly dependent.

c) (i) Given: $\mathbf{v}^1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v}^2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

(ii) Let's consider the linear equation system $\left[\mathbf{v}^1\ \mathbf{v}^2\right] \mathbf{x} = \mathbf{0}$

$$\begin{pmatrix} 1 & 0 & \vdots & 0 \\ 1 & 1 & \vdots & 0 \\ 0 & 1 & \vdots & 0 \end{pmatrix}$$

We know from the first row that $x_1 = 0$ and from the third row that $x_2 = 0$

(iii) Since $\mathbf{0}$ is the only solution to the linear equation syatem in (ii), we know that $\mathbf{v}^1$ and $\mathbf{v}^2$ are linearly independent.

d) (i) Given: $\mathbf{v}^1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v}^2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $\mathbf{v}^3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

(ii) Let's consider the linear equation system $\left[\mathbf{v}^1\ \mathbf{v}^2\ \mathbf{v}^3\right] \mathbf{x} = \mathbf{0}$

$$\begin{pmatrix} 1 & 0 & 1 & \vdots & 0 \\ 1 & 1 & 0 & \vdots & 0 \\ 0 & 1 & 1 & \vdots & 0 \end{pmatrix}$$

Add (-1) times the first row to the third row:

$$\begin{pmatrix} 1 & 0 & 1 & \vdots & 0 \\ 1 & 1 & 0 & \vdots & 0 \\ -1 & 1 & 0 & \vdots & 0 \end{pmatrix}$$

Add the third row to the second row:

$$\begin{pmatrix} 1 & 0 & 1 & \vdots & 0 \\ 0 & 2 & 0 & \vdots & 0 \\ -1 & 1 & 0 & \vdots & 0 \end{pmatrix}$$

Divide the second row by 2

$$\begin{pmatrix} 1 & 0 & 1 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 0 \\ -1 & 1 & 0 & \vdots & 0 \end{pmatrix}$$

Add (-1) times the second row to the third row:

$$\begin{pmatrix} 1 & 0 & 1 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 0 \\ -1 & 0 & 0 & \vdots & 0 \end{pmatrix}$$

Add the third row to the first row:

$$\begin{pmatrix} 0 & 0 & 1 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 0 \\ -1 & 0 & 0 & \vdots & 0 \end{pmatrix}$$

We know from the first row that $x_3 = 0$, from the second row that $x_2 = 0$, and from the third row that $x_1 = 0$

(iii) Since $\mathbf{0}$ is the only solution to the linear equation syatem in (ii), we know that $\mathbf{v}^1$, $\mathbf{v}^2$ and $\mathbf{v}^3$ are linearly independent.

Problem 3

Simon & Blume question 11.9 (page 246).

Ans: This questions asks you to do a basis transformation, i.e., write a vector **b** as a linear combination of vectors.

a) (i) Given: $\mathbf{v}^1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{v}^2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$,

(ii) Let's consider the linear equation system $\left[\mathbf{v}^1\ \mathbf{v}^2\right] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 & \vdots & 2 \\ 2 & 4 & \vdots & 2 \end{pmatrix}$$

Add (-2) times the first row to the second row:

$$\Leftrightarrow \begin{pmatrix} 1 & 1 & \vdots & 2 \\ 0 & 2 & \vdots & -2 \end{pmatrix}$$

Divide the second row by 2:

$$\Leftrightarrow \begin{pmatrix} 1 & 1 & \vdots & 2 \\ 0 & 1 & \vdots & -1 \end{pmatrix}$$

Add (-1) times the second row to the first:

$$\Leftrightarrow \begin{pmatrix} 1 & 0 & \vdots & 3 \\ 0 & 1 & \vdots & -2 \end{pmatrix}$$

(iii) We know from (ii) that **b** can be written as $\mathbf{b} = 3\ \mathbf{v}^1$ - $\mathbf{v}^2$

d) (i) Given: $\mathbf{v}^1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v}^2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{v}^3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$,

(ii) consider the equation system $\left[\mathbf{v}^1\ \mathbf{v}^2\ \mathbf{v}^3\right] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 & 0 & \vdots & 1 \\ 1 & 0 & 1 & \vdots & 2 \\ 0 & 1 & 1 & \vdots & 3 \end{pmatrix}$$

Add (-1) times the first row to the second row:

$$\begin{pmatrix} 1 & 1 & 0 & \vdots & 1 \\ 0 & -1 & 1 & \vdots & 1 \\ 0 & 1 & 1 & \vdots & 3 \end{pmatrix}$$

Add the second row to the third row:

$$\begin{pmatrix} 1 & 1 & 0 & \vdots & 1 \\ 0 & -1 & 1 & \vdots & 1 \\ 0 & 0 & 2 & \vdots & 4 \end{pmatrix}$$

Divide the third row by 2

$$\begin{pmatrix} 1 & 1 & 0 & \vdots & 1 \\ 0 & -1 & 1 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & 2 \end{pmatrix}$$

Add (-1) times the third row to the second row:

$$\begin{pmatrix} 1 & 1 & 0 & \vdots & 1 \\ 0 & -1 & 0 & \vdots & -1 \\ 0 & 0 & 1 & \vdots & 2 \end{pmatrix}$$

Multiply the third row by (-1)

$$\begin{pmatrix} 1 & 1 & 0 & \vdots & 1 \\ 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & 2 \end{pmatrix}$$

Add (-1) times the second row to the first row:

$$\begin{pmatrix} 1 & 0 & 0 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & 2 \end{pmatrix}$$

(iii) We know from (ii) that **b** can be written as $\mathbf{v}^2 + 2\ \mathbf{v}^3$

Problem 4

Simon & Blume question 11.14 (page 249). Explain your answer.

Ans: This question asks you to determine whether a set of vectors is a basis or not. From Simon & Blume Theorem 11.7 (page 248) you know that a basis of $\mathbb{R}^n$ contains exactly n vectors and from Theorem 11.8 you know that they form a basis iff the n vectors are linearly independent. There are again several ways to do this.

(i) Following problem 2, show that 3 vectors are linearly independent.
(ii) Calculate the determinant of the 3 vectors in part b)-d). If it is nonzero, we know that the 3 vectors are linearly independent and therefore form a basis

Since we did method (i) in problem 3, let's do method (ii) here.

a) Two vectors cannot span $\mathbb{R}^3$ and we therefore know that the vectors can't be a basis. (In general, a set of n vectors can span at most a space of dimension n).

b) det $= \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{vmatrix}$

Develop after the third column:

det $= 1 \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} =$ 1 - 2 + 2 -1 = 0

Since the detrminant is zero, the three vectors are linearly dependent and thus can't be a basis.

c) det $= \begin{vmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \\ 9 & 8 & 7 \end{vmatrix}$

Develop after the third column:

det $= 4 \begin{vmatrix} 3 & 2 \\ 9 & 8 \end{vmatrix} - 1 \begin{vmatrix} 6 & 5 \\ 9 & 8 \end{vmatrix} + 7 \begin{vmatrix} 6 & 5 \\ 3 & 2 \end{vmatrix} =$ 4(24-18) - (48-45) + 7(12-15) = 24-3-21 = 0

Since the detrminant is zero, the three vectors are linearly dependent and thus can't be a basis.

d) det $= \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{vmatrix}$

Develop after the third column:

det $= 1 \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} =$ 1 - 2 = -1

Since the detrminant is nonzero, the three vectors are linearly independent and span $\mathbb{R}^3$ and hence are a basis.

e) 4 vectors can not be a basis for $\mathbb{R}^3$ as they have to be linearly dependent. (In general, if your vector space has dimension n and you have more than n vectors, they have to be linearly dependent).

Problem 5

Consider the following conjecture: Let V be a vector space spanned by the set $\{\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_n\}$. The set $\{\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_n\}$ is a minimal spanning set for V iff the vectors $\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_n$ are linearly independent vectors. Is the conjecture true? Prove your answer.

Ans: The conjecture is false because of the "$\Leftarrow$": Consider:

$$\mathbf{v}^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

And let $V = \{\alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \alpha \in \mathbb{R}\}$ (A line in the plane where $x_3 = 0$).

Clearly, $\mathbf{v}^1, \mathbf{v}^2$, and $\mathbf{v}^3$ are linearly independent, they span V (As they span $\mathbb{R}^3$ and hence any subset of it) but they are *not* a minimal spanning set (as you can drop $\mathbf{v}^3$ and still span V).

Problem 6

Let U and W be vector subspaces of the vectorspace V. Define the space

$U + W = \{\mathbf{x} | \mathbf{x} = \mathbf{u} + \mathbf{w}, \mathbf{u} \in U, \mathbf{w} \in W\}$ Show that $U + W$ is a vector subspace of V.

Ans: Recall that you have to check three conditions to show that X is a vector subspace of the vetor space V:

(i) X has to be nonempty
(ii) $\forall \mathbf{x}^1, \mathbf{x}^2 \in X : \mathbf{x}^1 + \mathbf{x}^2 \in X$
(iii) $\forall \mathbf{x}^1 \in X, \alpha \in \mathbb{R}_+ : \alpha \mathbf{x}^1 \in X$

Now let's prove the claim that if U, W are vector subspaces then so is $U + W$.

(i) Since $\mathbf{0}$ is an element of any vector subspace it is an element of U and W. But therefore

$$\mathbf{0} = \underbrace{\mathbf{0}}_{\in U} + \underbrace{\mathbf{0}}_{\in W}$$

and hence $\mathbf{0}$ is an element of $U + W$. Consequently, $U + W$ is nonempty.

(ii) (1) $\forall \mathbf{x}^1, \mathbf{x}^2 \in U + W : \exists \mathbf{u}^1, \mathbf{u}^2 \in U$ and $\exists \mathbf{w}^1, \mathbf{w}^2 \in W$ such that $\mathbf{x}^1 = \mathbf{u}^1 + \mathbf{w}^1$ and $\mathbf{x}^2 = \mathbf{u}^2 + \mathbf{w}^2$.
(2) Since U is a vector space we know that $\mathbf{u}^1 + \mathbf{u}^2 \in U$
(3) Since W is a vector space we know that $\mathbf{w}^1 + \mathbf{w}^2 \in W$
(4) From (2) and (3) we therfore know that
$\mathbf{x}^1 + \mathbf{x}^2 = \mathbf{u}^1 + \mathbf{w}^1 + \mathbf{u}^2 + \mathbf{w}^2$ (By (1)) $= \underbrace{\mathbf{u}^1 + \mathbf{u}^2}_{\in U} + \underbrace{\mathbf{w}^1 + \mathbf{w}^2}_{\in W}$
is an element of $U + W$

(iii) (1) $\forall \mathbf{x}^1 \in U + W : \exists \mathbf{u}^1 \in U$ and $\exists \mathbf{w}^1 \in W$ such that $\mathbf{x}^1 = \mathbf{u}^1 + \mathbf{w}^1$
(2) Since U is a vector space we know that $\forall \alpha \in \mathbb{R}_+ : \alpha \mathbf{u}^1 \in U$
(3) Since W is a vector space we know that $\forall \alpha \in \mathbb{R}_+ : \alpha \mathbf{w}^1 \in W$
(4) From (2) and (3) we therfore know that
$\alpha \mathbf{x}^1 = \alpha(\mathbf{u}^1 + \mathbf{w}^1)$ (By (1))
$= \underbrace{\alpha \mathbf{u}^1}_{\in U} + \underbrace{\alpha \mathbf{w}^1}_{\in W}$
is an element of $U + W$

Problem 7

a) Let A be a symmetric (nxn) matrix with one or more negative eigenvalues. What can you say about the determinant of the matrix and its rank?

b) Let A be symmetric (2x2) matrix. Is it true that A is indefinite iff its determinant is negative. Explain your answer (No formal proof necessary).

Ans:

a) *Let A be a symmetric (nxn) matrix with one or more negative eigenvalues. What can you say about the determinant of the matrix and its rank?*
The determinant of a symmetric matrix is the product of its eigenvalues. Therefore, the determinant is
   (i) *negative* if it has an odd number of negative eigenvalues (and all eigenvalues are different from zero).
   (ii) *positive* if it has an even number of negative eigenvalues (and all eigenvalues are different from zero).
   (iii) *zero* if it has at least one eigenvalue that is zero.

   The number of negative eigenvalues doesn't say anything about the rank of a matrix. From (iii) above you know that the rank is of full rank iff the matrix has only nonzero eigenvalues.

b) *Let A be symmetric (2x2) matrix. Is it true that A is indefinite iff its determinant is negative?*
We know that every symmetric (2x2) matrix has exactly 2 real eigenvalues. Let's consider all possible cases:
   (i) Both eigenvalues are positive $\Rightarrow$ the matrix is positive definite
   (ii) Both eigenvalues are negative $\Rightarrow$ the matrix is negative definite
   (iii) One eigenvalues is positive, the other is zero $\Rightarrow$ the matrix is positive semi-definite
   (iv) One eigenvalues is negative, the other is zero $\Rightarrow$ the matrix is negative semi- definite
   (v) Both eigenvalues are zero $\Rightarrow$ the matrix is both postive semi-definite and negative semi-definite
   (vi) One eigenvalue is positive, the other one is negative $\Rightarrow$ the matrix is indefinite

   Hence, A is indefinite iff its determinant is negative. (The product of its two eigenvalues is negative).