PROBLEM SET #04- ANSWER KEY

FIRST LIN ALGEBRA PROBLEM SET

(1) Simon & Blume question 10.5 (page 208).

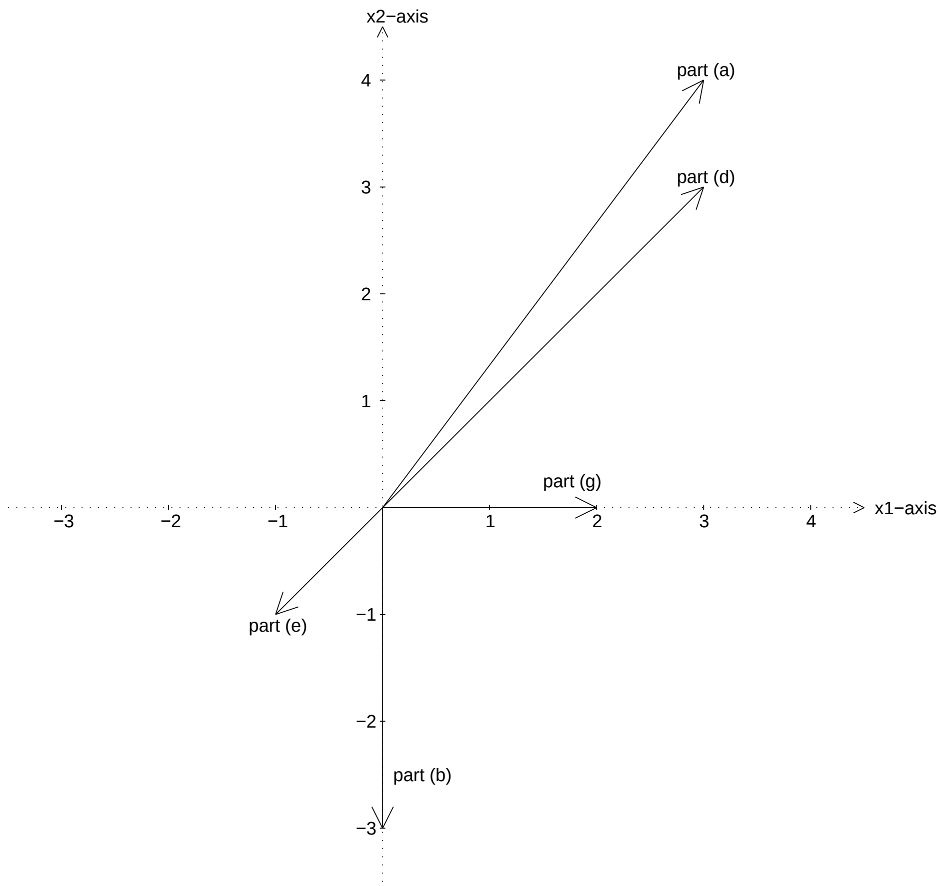
Ans: Let **u** = (1,2), **v** = (0,1), **w** = (1,-3), **x** = (1,2,0) and **z** = (0,1,1)

(i) **u**+**v** = (1,2)+(0,1) = (1,3).
(ii) -4***w** = -4*(1,-3) = (-4,12).
(iii) **u**+**z** is not defined as **u** is an (1x2) vector and **z** is an (1x3) vector.
(iv) 3***z** = 3*(0,1,1) = (0,3,3).
(v) 2***v** = 2*(0,1) = (0,2).
(vi) **u**+2***v** = (1,2)+2*(0,1) = (1,4).
(vii) **u**-**v** = (1,2)-(0,1) = (1,1).
(viii) 3***x**+**z** = 3*(1,2,0)+(0,1,1) = (3,7,1).
(ix) -2***x** = -2*(1,2,0) = (-2,-4,0).
(x) **w** + 2***x** is not defined as **w** is an (1x2) vector and **x** is an (1x3) vector

(2) Simon & Blume question 10.10 (page 220)

Ans: The vectors for part a,b,d,e, and g are displayed in figure 1. The vectors for part c and f are diplayed in figure 2.

FIGURE 1. The vectors for part a,b,d,e, and g



a) $||(3,4)|| = \sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$
b) $||(0,-3)|| = \sqrt{0^2+(-3)^2} = \sqrt{0+9} = \sqrt{9} = 3$
c) $||(1,1,1)|| = \sqrt{1^2+1^2+1^2} = \sqrt{1+1+1} = \sqrt{3}$
d) $||(3,3)|| = \sqrt{3^2+3^2} = \sqrt{9+9} = \sqrt{2*9} = 3*\sqrt{2}$
e) $||(-1,-1)|| = \sqrt{(-1)^2+(-1)^2} = \sqrt{1+1} = \sqrt{2}$
f) $||(1,2,3)|| = \sqrt{1^2+2^2+3^2} = \sqrt{1+4+9} = \sqrt{14}$
g) $||(2,0)|| = \sqrt{2^2+0^2} = \sqrt{4+0} = \sqrt{4} = 2$
h) $||(1,2,3,4)|| = \sqrt{1^2+2^2+3^2+4^2} = \sqrt{1+4+9+16} = \sqrt{30}$
i) $||(3,0,0,0)|| = \sqrt{3^2+0^2+0^2+0^2} = \sqrt{9+0+0+0} = \sqrt{9} = 3$

(3) Suppose you know that the angle between two vectors is as given below. What do you know about the sign of the inner product of the two vectors?

a) 180

b) 53

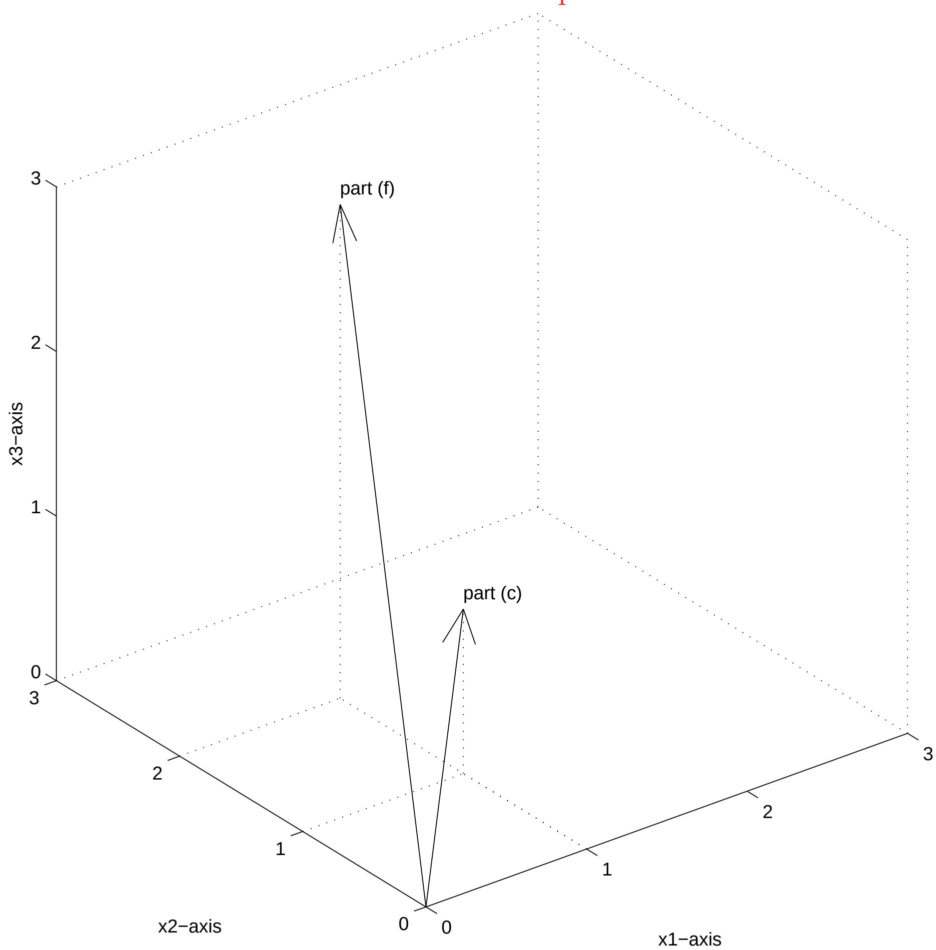
c) 320

d) 90

Ans: Recall that the sign of the inner product of two vectors v_1 and v_2 is given by the sign of the cosinus of the angle between them.

a) $cos(180) = -1 \Rightarrow$ The sign of the inner product is negative.

FIGURE 2. The vectors for part 2c and 2f



b) $cos(53) > 0 \Rightarrow$ The sign of the inner product is positive.
c) $cos(320) > 0 \Rightarrow$ The sign of the inner product is positive.
d) $cos(90) = 0 \Rightarrow$ The sign of the inner product is zero as the vectors are perpendicular.

(4) Using the definition of linear independency,i.e.,

a set of vectors $\{v^1, ...v^k..., v^m\}$ is a *linear independent set* if for all $\mathbf{t} \in \mathbb{R}^m$, $\sum_{k=1}^m t_k v^k = 0$ implies $\mathbf{t} = 0$, prove the following properties: (Note: once you have shown a property, you can use it to show the following ones)

a) A singleton vector is a linear independent set *if and only if* it is not the zero vector.

Ans: First, let's prove the sufficiency condition by contrapositive ($(notB \Rightarrow notA) \Leftrightarrow (A \Rightarrow B)$). The nul vector is not a linear independent set, because we can find a $\lambda \neq 0$ such that $\lambda.v = 0$ (take $\lambda = 1$ for example). That proves that a singleton vector is a linear independent set then it is not equal to zero. For the necessary condition, lets take a non zero vector v, and let λ be a scalar such that $\lambda.v = 0$. Knowing that $v \neq 0$ there is at least one non zero component v_l of v. For this component, we can write $\lambda = 0/v_l$ so $\lambda = 0$ and v is a linear independent set.

b) Two nonzero vectors are linearly independent *if and only if* they are not colinear (or proportional, i.e. for two vectors (u, v),there exists $\lambda \in \mathbb{R}$ such that $u = \lambda.v$)

Ans: First, let's prove the sufficiency condition by contradiction. Let (u, v) be a couple of nonzero linearly independent vector that are colinear. By definition of colinearity, there exist a scalars α such that $v = \alpha.u$. Thus $v - \alpha.u = 0$, and this constitutes a linear combination of (u, v) equal to zero but with non zero scalars (at least the coefficient 1 in front of v is not equal to zero). Thus (u, v) is linear dependent which contradicts the assumption.
Second, we prove the necessary part by contrapositive ($(notB \Rightarrow notA) \Leftrightarrow (A \Rightarrow B)$). Suppose that u and v are two linearly dependent non-zero vectors. Then we just need to show that they are colinear. By definition of linear dependence, there exist two scalars $(\alpha, \beta) \neq (0, 0)$ such that $\alpha.u + \beta.v = 0$. One of these scalars is necessarily different from zero, suppose it is α. Then, we can write $u = -\frac{\beta}{\alpha}.v$, so these two vectors are colinear. This proves that two vector that are not colinear are linearly independent .

c) for $n > 1$, $(v^1, ..., v^n)$ is a linear dependent set *if and only if* one of the vector in the set v^i is a linear combination of the other $n - 1$ vectors.

Ans: The case of two vectors was shown in b). Now lets look at a case where $n > 2$. Suppose $(v^1, ..., v^n)$ are linearly dependent then there exist a set of scalars $(\lambda_1, ..., \lambda_n)$ not all equal to

zero such that $\sum_{k=1}^{n} \lambda_k.v^k = 0$. There is at least one scalar not equal to zero, suppose $\lambda_i \neq 0$ then we have $\lambda_i.v^i = -\sum_{k\neq i} \lambda_k.v^k$ or $v^i = -\frac{1}{\lambda_i}\sum_{k\neq i} \lambda_k.v^k$ so v^i is a linear combination of the others $n-1$ vectors with coefficients $\mu_k = -\frac{\lambda_k}{\lambda_i}$. This shows the sufficiency part.
Conversely, if there exists $i \in [|0,n|]$ such that v^i is a linear combination of the $n-1$ other vectors, there exists a set of scalars $(\mu_1, ..., \mu_{i-1}, \mu_{i+1}, ..., \mu_n)$ such that $v^i = \sum_{k\neq i} \mu_k.v^k$. We will construct a set of scalars in order to prove that these vectors are linearly dependent: set $\lambda_k = \mu_k$ for $k \neq i$ and $\lambda_i = -1$.
Then we have that $\sum_{k\neq i} \mu_k.v^k - v^i = 0$, i.e. $\sum_{k\neq i} \lambda_k.v^k + \lambda_i v^i = 0$, which simplifies to $\sum_{k=1}^{n} \lambda_k.v^k = 0$. So we found a linear combination of the vectors equal to zero for which at least one scalar $\lambda_i = -1$ is not equal zero, so the vectors are linearly dependent.

d)If $(v^1, ..., v^n)$ is a linear independent set, and y is a different vector, $(v^1, ..., v^n, y)$ is linearly dependent *if and only if* y is a linear combination of $v^1, ..., v^n$

Ans: First we look at the sufficiency case. Suppose $(v^1, ..., v^n)$ is a linear independent set, and $(v^1, ..., v^n, y)$ is linearly dependent. Because of this linear dependence, there exists a set of scalars $(\lambda_1, ..., \lambda_{n+1})$ not all equal to 0 such that $\sum_{k=1}^{n} \lambda_k.v^k + \lambda_{n+1}y = 0$. There are two possibilities: $\lambda_{n+1}y = 0$ or $\lambda_{n+1}y \neq 0$.
- If $\lambda_{n+1} = 0$ then $\sum_{k=1}^{n} \lambda_k.v^k = 0$ and by linear independence all the scalars are equal to zero which is impossible by definition of the linear dependence of $(v^1, ..., v^n, y)$. Thus $\lambda_{n+1} \neq 0$. Knowing that, we can write that $\sum_{k=1}^{n} \lambda_k.v^k = -\lambda_{n+1}y$ and $y = -\frac{1}{\lambda_{n+1}}\sum_{k=1}^{n} \lambda_k.v^k$. Thus y is a linear combination of the n vectors with coefficients $-\frac{\lambda_k}{\lambda_{n+1}}$.
For the necessary part, suppose that y is a linear combination of $(v^1, ..., v^n)$, then there exists a set of scalars $(\mu_1, ..., \mu_n)$ such that $y = \sum_{k=1}^{n} \mu_k.v^k$. By a similar reasoning as in our proof of c) we can construct a set of scalars $\lambda_1, ..., \lambda_{n+1}$not equal to zero such that $\sum_{k=1}^{n} \lambda_k.v^k + \lambda_{n+1}y = 0$. We just need to define these scalars: set $\lambda_k = \mu_k$ for all $k \in [|1,n|]$ and set $\lambda_{n+1} = -1$. Thus we just showed that $(v^1, ..., v^n, y)$ is linearly dependent.

e) If $(v^1, ..., v^n)$ is a linear independent set, then any subset of this set (such as $v^1, ..., v^i$, with $i < n$) is also linear independent.

Ans: This case only requires to show one sufficiency condition, I will proceed by contradiction. Suppose that $(v^1, ..., v^n)$ is linearly independent but that $(v^1, ..., v^i)$ is linearly dependent. Then there exists a set of scalars $(\lambda_1, ..., \lambda_i)$ not all equal to zero such that $\sum_{k=1}^{i} \lambda_k.v^k = 0$. We can add zero to this expression, so for $k \in \{i+1, ..., n\}$ set $\lambda_k = 0$ then we have: $\sum_{k=1}^{i} \lambda_k v^k + \sum_{k=i+1}^{n} \lambda_k.v^k = 0$, i.e., $\sum_{k=1}^{n} \lambda_k v^k = 0$. But at least one of the scalars $\{\lambda_1, ..., \lambda_i\}$ is not equal to zero because $\mathbf{\Lambda} \neq \mathbf{0}$. This contradicts our assumption of linear independence of $(v^1, ..., v^n)$ so necessarily $(v^1, ..., v^i)$ is linearly independent. This is true for any subset of the original set of linear independent vectors.

(5) Show that if $\mathbf{v}^1$ and $\mathbf{v}^2$ are linearly independent vectors in $\mathbb{R}^2$, then any vector $\mathbf{w} \in \mathbb{R}^2$ can be written as a linear combination of $\mathbf{v}^1$ and $\mathbf{v}^2$. To show this, use *only* the fact that $\mathbf{v}^1$ and $\mathbf{v}^2$ are linear independent iff they are not colinear. (i.e., do not use properties of matrices or determinants.)

Hint #1: Try to write $\mathbf{w}$ as $\alpha\mathbf{v}^1 + \beta\mathbf{v}^2$. Under what conditions can you solve for α and β?

Hint #2: one way of utilizing hint #1 is to use the following rather clumsy characterization of colinearity:

$$\mathbf{v}^1 \text{ and } \mathbf{v}^2 \text{ are colinear iff } \begin{cases} \text{neither } v_1^1 \text{ nor } v_1^2 \text{ is zero} \Longrightarrow \frac{v_2^1}{v_1^1} = \frac{v_2^2}{v_1^2} & \text{or} \\ \text{neither } v_2^1 \text{ nor } v_2^2 \text{ is zero} \Longrightarrow \frac{v_1^1}{v_2^1} = \frac{v_1^2}{v_2^2} & \text{or} \\ \text{at least one of the vectors is zero} \end{cases} \tag{1}$$

The second and third branches of this characterization say, respectively, that either "rise over run" or "run over rise" must be the same for the two vectors.

Ans: Following hint #1, we will write

$$w_1 = \alpha v_1^1 + \beta v_1^2 \tag{2a}$$

$$w_2 = \alpha v_2^1 + \beta v_2^2 \tag{2b}$$

and will attempt to solve for α and β. We'll start with the "if" part, i.e., assume linear independence and show that we can solve for α and β. Since $\mathbf{v}^1$ and $\mathbf{v}^2$ are linearly independent and hence not colinear, either (a) $v_1^1 \neq 0$ or (b) $v_2^1 \neq 0$ (i.e., $\mathbf{v}$ cannot be zero). Without loss of generality, assume that (a) is true. (If (b) is true, flip the coefficients.) We can then divide both sides of (2a) by v_1^1 and obtain

$$\alpha = \frac{w_1}{v_1^1} - \beta\frac{v_1^2}{v_1^1} \tag{3}$$

now, subsituting for α in (2a)

$$w_2 = \left(\frac{w_1}{v_1^1} - \beta\frac{v_1^2}{v_1^1}\right)v_2^1 + \beta v_2^2$$

so that

$$w_2 v_1^1 = w_1 v_2^1 + \beta\left(v_2^2 v_1^1 - v_1^2 v_2^1\right) \tag{4}$$

We can now solve for β iff $\left(v_1^2 v_2^1 - v_2^2 v_1^1\right) \neq 0$, in which case

$$\beta = \frac{w_2 v_1^1 - w_1 v_2^1}{v_2^2 v_1^1 - v_1^2 v_2^1}$$

and so, substituting this expression for β back into (3)

$$\begin{aligned}
\alpha &= \frac{w_1}{v_1^1} - \beta\frac{v_1^2}{v_1^1} \\
&= \frac{w_1}{v_1^1} - \left(\frac{w_2 v_1^1 - w_1 v_2^1}{v_2^2 v_1^1 - v_1^2 v_2^1}\right)\frac{v_1^2}{v_1^1} \\
&= \frac{w_1\left(v_2^2 v_1^1 - v_1^2 v_2^1\right) - v_1^2\left(w_2 v_1^1 - w_1 v_2^1\right)}{v_1^1\left(v_2^2 v_1^1 - v_1^2 v_2^1\right)} \\
&= \frac{w_1 v_2^2 - v_1^2 w_2}{v_2^2 v_1^1 - v_1^2 v_2^1}.
\end{aligned}$$

We now need to show that $\mathbf{v}^1$ and $\mathbf{v}^2$ being linearly independent implies $\left(v_1^2 v_2^1 - v_2^2 v_1^1\right) \neq 0$. First note that if $v_1^2 = 0$, then necessarily $v_2^2 \neq 0$ and hence $\left(v_1^2 v_2^1 - v_2^2 v_1^1\right) = v_2^2 v_1^1 \neq 0$. On the other hand, if $v_1^2 \neq 0$ then $\left(v_1^2 v_2^1 - v_2^2 v_1^1\right) = v_1^1 v_1^2\left(\frac{v_2^1}{v_1^1} - \frac{v_2^2}{v_1^2}\right)$. Now $v_1^1 v_1^2$ is non-zero by assumption, and we obtain that the term in parentheses is non-zero by negating the implication in the second branch of our characterization of colinearity.
Now for the "only if" part. Assume that $\mathbf{v}^1$ and $\mathbf{v}^2$ are linearly dependent. We need to show that we can't solve for α and β. From β, this will be the case if $\left(v_2^2 v_1^1 - v_1^2 v_2^1\right) = 0$. From (1), there are three possibilities:

(a) for $i = 1, 2$, $v_1^i = v_2^i = 0$,
(b) both v_l^1 and v_1^2 are nonzero *and* $\frac{v_2^1}{v_1^1} = \frac{v_2^2}{v_1^2}$.
(c) both v_2^1 and v_2^2 are nonzero *and* $\frac{v_1^1}{v_2^1} = \frac{v_1^2}{v_2^2}$.

All three conditions imply that $\left(v_2^2 v_1^1 - v_1^2 v_2^1\right) = 0$.