Fall 2008 **ARE211**

Problem Set #09

Char of Functions Problem Set

Due date: Nov 13

Problem 1

Show that a function $f : X \to A$ is invertible $\Leftrightarrow$ f is bijective

Problem 2

Which of the following functions is bijective? If it is bijective, give the inverse.

a) $f : \mathbb{R} \to \mathbb{R}$ $f(x) = 2x + 4$

b) $f : \mathbb{R} \to \mathbb{R}$ $f(x) = |2x + 4|$

c) $f : [-2, \infty) \to \mathbb{R}$ $f(x) = |-2x - 4|$

d) $f : [-2, \infty) \to [0, \infty)$ $f(x) = |-2x - 4|$

e) $f : (0, \infty) \to (0, \infty)$ $f(x) = \begin{cases} 1/x & \text{if } x \text{ rational} \\ x & \text{if } x \text{ irrational} \end{cases}$

Problem 3

Let **A** be a (nxn) matrix. The main diagonal consists of the elements $\{a_{ii}, i = 1..n\}$. Show that

a) A necessary condition for **A** to be positive definite is that all elements on the main diagonal are strictly positive.

b) A necessary condition for **A** to be negative definite is that all elements on the main diagonal are strictly negative.

c) A sufficient condition for **A** to be indefinite is that at least one element on the main diagonal is strictly positive and at least one element on the main diagonal is strictly negative.

Problem 4

For each of the following matrices state whether they are (i) positive definite, (ii) positive semi-definite, (iii) negative semi-definite, (iv) negative definite, or (v) indefinite (dependent on the values of the parameter α, β, γ).

a) $\mathbf{A}_\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 1 \\ 3 & 1 & \alpha \end{pmatrix}$

b) $\mathbf{B}_{\alpha,\beta,\gamma} = \begin{pmatrix} 1 & 2 & \beta \\ 2 & -3 & \alpha \\ \beta & \alpha & \gamma \end{pmatrix}$

c) $\mathbf{C}_{\alpha,\beta} = \begin{pmatrix} \alpha & 1 & 2 \\ 1 & \beta & 2 \\ 2 & 2 & 4 \end{pmatrix}$

Problem 5

Which of the following function is homogeneous? If it is homogeneous, give the degree of homogeneity.

a) $f(x, y) = 3x^5y + 2x^2y^4 - 3x^3y^3$

b) $f(x, y) = 3x^5y + 2x^2y^4 - 3x^3y^4$

c) $f(x, y) = 3\frac{\sqrt{x}}{\sqrt{y}} + 2\frac{x}{y} + 7$

d) $f(x, y) = \frac{(x^2-y^2)}{(x^2+y^2)} + 6$

Problem 6

In the following consider functions that map from the same domain to the same range.

a) Show that the product of two homogeneous functions is homogeneous again.

b) Show that the sum of two homogenous function is homogeneous if they are homogeneous of the same degree.

c) Is it possible that two functions f and g are both homogeneous of degree k and the sum f+g is homogeneous of a degree different from k? If yes, give an example, if not, explain why.

Problem 7

a) Prove Euler's theorem (Hint: Simon & Blume).

b) Prove that if $f(x_1, x_2)$ is a twice continously differentiable function that is homogeneous of degree k then

$$x_1^2 f_{x_1x_1} + 2x_1x_2 f_{x_1x_2} + x_2^2 f_{x_2x_2} = k(k-1)f$$