Fall 2008 **ARE211**

PROBLEM SET #09

CHAR OF FUNCTIONS PROBLEM SET

DUE DATE: NOV 13

Problem 1

Show that a function $f : X \to A$ is invertible $\Leftrightarrow$ f is bijective

Ans: Recall the following definitions:

(A) A function is injective if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.
(B) A function is surjective if $\forall a \in A\ \exists x \in X$ s.t. $f(x) = a$.
(C) A function $f : X \to A$ is invertible if $\exists f^{-1} : A \to X$ sucht that $\forall a \in A, f(f^{-1}(a)) = a$ and $\forall x \in X, f^{-1}(f(x)) = x$.

We need to show $C \Leftrightarrow (A \wedge B)$.

"$\Rightarrow$" Let's prove the contrapositive, i.e. $(\neg A \vee \neg B) \Rightarrow \neg C$

case 1 If f is not injective then $\exists x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$. Assume there exists an inverse f^{-1}. We know from (C) that $x_1 = f^{-1}(f(x_1)) = f^{-1}(f(x_2)) = x_2$ which contradicts $x_1 \neq x_2$. Hence f can't be invertible.

case 2 If f is not surjective then $\exists \bar{a} \in A$ with $\forall x \in X,\ f(x) \neq \bar{a}$. However, since f^{-1} maps from A to X, we know that $f^{-1}(\bar{a}) \in X$ and hence $f(f^{-1}(\bar{a})) \neq a$. Hence f can't be invertible.

"$\Leftarrow$" Since f is surjective we know that $\forall a \in A\ \exists x(a) \in X$ s.t. $f(x) = a$.
Furthermore, since f is injective we know that $x(a)$ is unique as $a = f(x_1) = f(x_2)$ implies $x_1 = x_2$. Let's define f^{-1} as x(a). Clearly, $\forall a \in A, f(f^{-1}(a)) = a$ and $\forall x \in X, f^{-1}(f(x)) = x$ by the definition of x(a) and consequently f is invertible.

Problem 2

Which of the following functions is bijective? If it is bijective, give the inverse.

a) $f : \mathbb{R} \to \mathbb{R}\ f(x) = 2x + 4$

b) $f : \mathbb{R} \to \mathbb{R}$ $f(x) = |2x + 4|$

c) $f : [-2, \infty) \to \mathbb{R}$ $f(x) = |-2x - 4|$

d) $f : [-2, \infty) \to [0, \infty)$ $f(x) = |-2x - 4|$

e) $f : (0, \infty) \to (0, \infty)$ $f(x) = \begin{cases} 1/x & \text{if } x \text{ rational} \\ x & \text{if } x \text{ irrational} \end{cases}$

Ans:

Recall the following definitions:

(I) A function is injective if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.
(S) A function is surjective if $\forall a \in A\ \exists x \in X$ s.t. $f(x) = a$.

a) $f : \mathbb{R} \to \mathbb{R}\ f(x) = 2x + 4$
   (I) $f(x_1) = f(x_2) \Leftrightarrow 2x_1 + 4 = 2x_2 + 4 \Leftrightarrow x_1 = x_2$.
   Hence f is injective
   (S) $\forall a \in \mathbb{R} : \exists x(a) = \frac{a-4}{2} \in \mathbb{R}$ s.t. $f(x(a)) = 2\frac{a-4}{2} + 4 = a - 4 + 4 = a$.
   Hence f is surjective.
   $\Rightarrow$ The function is bijective and the inverse is $f^{-1}(a) = \frac{a-4}{2}$.
b) $f : \mathbb{R} \to \mathbb{R}\ f(x) = |2x + 4|$
   (I) $f(x_1 = -3) = |2(-3) + 4| = |-6 + 4| = 2 = |-2 + 4| = |2(-1) + 4| = f(x_2 = -1)$ and $x_1 \neq x_2$.
   Hence f is *not* injective
   $\Rightarrow$ The function is *not* bijective.
c) $f : [-2, \infty) \to \mathbb{R}\ f(x) = |-2x - 4|$
   (I) $a = -1 \in \mathbb{R}$ but since the absolute value is greater or equal than zero, $\forall x \in [-2, \infty) :$ $f(x) \geq 0 > -1 = a$
   Hence f is *not* surjective
   $\Rightarrow$ The function is *not* bijective.
d) $f : [-2, \infty) \to [0, \infty)\ f(x) = |-2x - 4| = |2x + 4|$
   Note: Since $x \geq -2$ we know $2x \geq -4$ and hence $2x + 4 \geq 0$. Hence we can write $f(x) = |2x + 4| = 2x + 4$
   (I) $f(x_1) = f(x_2) \Leftrightarrow 2x_1 + 4 = 2x_2 + 4 \Leftrightarrow x_1 = x_2$.
   Hence f is injective
   (S) $\forall a \in [0, \infty) : \exists x(a) = \frac{a-4}{2} \in [-2, \infty)$ s.t. $f(x(a)) = 2\frac{a-4}{2} + 4 = a$.
   Hence f is surjective.
   $\Rightarrow$ The function is bijective and the inverse is $f^{-1}(a) = \frac{a-4}{2}$.
e) $f : (0, \infty) \to (0, \infty)\ f(x) = \begin{cases} 1/x & if x rational \\ x & if x irrational \end{cases}$
   Note: If $x > 0$ is rational than so is $1/x$. Hence $f(x)$ is rational iff x is rational
   (I) If $f(x_1)$ is rational than $f(x_1) = f(x_2) \Leftrightarrow \frac{1}{x_1} = \frac{1}{x_2} \Leftrightarrow x_1 = x_2$.
   If $f(x_1)$ is irrational then $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$
   Hence f is injective.
   (S) $\forall a \in (0, \infty), a$ rational: $\exists x(a) = \frac{1}{a} \in (0, \infty)$ s.t. $f(x(a)) = \frac{1}{\frac{1}{a}} = a$.
   $\forall a \in (0, \infty), a$ irrational: $\exists x(a) = a \in (0, \infty)$ s.t. $f(x(a)) = f(a) = a$.
   Hence f is surjective.
   $\Rightarrow$ The function is bijective and the inverse is $f^{-1}(a) = f(a)$.

Problem 3

Let **A** be a (nxn) matrix. The main diagonal consists of the elements $\{a_{ii}, i = 1..n\}$. Show that

a) A necessary condition for **A** to be positive definite is that all elements on the main diagonal are strictly positive.

b) A necessary condition for **A** to be negative definite is that all elements on the main diagonal are strictly negative.

c) A sufficient condition for **A** to be indefinite is that at least one element on the main diagonal is strictly positive and at least one element on the main diagonal is strictly negative.

Ans: Let **A** be a (nxn) matrix.

a) *A necessary condition for* **A** *to be positive definite is that all elements on the main diagonal are strictly positive.* $\underline{Proof}$:
   (1) Assume that not all elements on the main diagonal are positive, i.e. $\exists k \in \{1, 2, ..., n\}$ s.t. $a_{kk} \leq 0$.
   (2) Consider the unit column vector $\mathbf{e}_k$ (which is 1 in the kth row and zero otherwise). $\mathbf{e}_k\mathbf{A}\mathbf{e}_k = 1 * a_{kk} * 1 = a_{kk} \leq 0$
   (3) From (2) we know that $\exists \mathbf{x} \neq 0$ such that $\mathbf{xAx} \leq 0$. Hence **A** can't be positive definite.

b) *A necessary condition for* **A** *to be negative definite is that all elements on the main diagonal are strictly negative.* $\underline{Proof}$:
   (1) Assume that not all elements on the main diagonal are positive, i.e. $\exists k \in \{1, 2, ..., n\}$ s.t. $a_{kk} \geq 0$.
   (2) Consider the unit column vector $\mathbf{e}_k$ (which is 1 in the kth row and zero otherwise). $\mathbf{e}_k\mathbf{A}\mathbf{e}_k = 1 * a_{kk} * 1 = a_{kk} \geq 0$
   (3) From (2) we know that $\exists \mathbf{x} \neq 0$ such that $\mathbf{xAx} \geq 0$. Hence **A** can't be negative definite.

c) *A sufficient condition for* **A** *to be indefinite is that at least one element on the main diagonal is strictly positive and at least one element on the main diagonal is strictly negative.* $\underline{Proof}$:
   (1) You are given that one element on the main diagonal is strictly positive and one is strictly negative, i.e. $\exists i, j \in \{1, 2, ..., n\}$ s.t. $a_{ii} > 0, a_{jj} < 0$.
   (2) Consider the unit column vector $\mathbf{e}_i$ (which is 1 in the ith row and zero otherwise). $\mathbf{e}_i\mathbf{A}\mathbf{e}_i = 1 * a_{ii} * 1 = a_{ii} > 0$

(3) Consider the unit column vector $\mathbf{e}_j$ (which is 1 in the jth row and zero otherwise). $\mathbf{e}_i\mathbf{A}\mathbf{e}_j = 1 * a_{jj} * 1 = a_{jj} < 0$

(4) From (2) and (3) we know that $\exists \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ such that $\mathbf{x}_1\mathbf{A}\mathbf{x}_1 > 0$ and $\mathbf{x}_2\mathbf{A}\mathbf{x}_2 < 0$. Hence $\mathbf{A}$ is indefinite.

Problem 4

For each of the following matrices state whether they are (i) positive definite, (ii) positive semi-definite, (iii) negative semi-definite, (iv) negative definite, or (v) indefinite (dependent on the values of the parameter α, β, γ).

a) $\mathbf{A}_\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 1 \\ 3 & 1 & \alpha \end{pmatrix}$

b) $\mathbf{B}_{\alpha,\beta,\gamma} = \begin{pmatrix} 1 & 2 & \beta \\ 2 & -3 & \alpha \\ \beta & \alpha & \gamma \end{pmatrix}$

c) $\mathbf{C}_{\alpha,\beta} = \begin{pmatrix} \alpha & 1 & 2 \\ 1 & \beta & 2 \\ 2 & 2 & 4 \end{pmatrix}$

Ans:

a) The first order leading principal minor is: $1 > 0$.

The second order leading principal minor is: $\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 5 - 4 = 1 > 0$

The third order leading principal minor is: (develop after 3rd column)

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 1 \\ 3 & 1 & \alpha \end{vmatrix} = 3\begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + \alpha\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = -34 + \alpha$$

Therefore we know that

$\alpha > 34$ The matrix is positive definite as all leading principal minors are positive.

$\alpha = 34$ The matrix is positive semi-definite as *all principal minors* are greater or equal than zero (there are 4 more besides the *leading* principal minors above):

First order: $5 \geq 0$ and $\alpha = 34 \geq 0$.

Second order: $\begin{vmatrix} 1 & 3 \\ 3 & 34 \end{vmatrix} = 25 \geq 0$ and $\begin{vmatrix} 5 & 1 \\ 1 & 34 \end{vmatrix} = 169 \geq 0$

$\alpha < 34$ The matrix is indefinite as the first order leading principal minor is positive and the third order leading principal minor is negative.

b) Since the main diagonal includes both positive and negative elements we know from problem 3 that the matrix is indefinite for all values of the parameters $\alpha, \beta,$ and γ.

c) For this question it is easiest to multiply out $\mathbf{x}'\mathbf{C}\mathbf{x}$:

$$\mathbf{x}'\mathbf{C}\mathbf{x} = \alpha x_1^2 + 2x_1x_2 + 4x_1x_3 + \beta x_2^2 + 4x_2x_3 + 4x_3^2$$
$$= (\alpha - 1)x_1^2 + (\beta - 1)x_2^2 + (x_1 + x_2 + 2x_3)^2$$

There are three cases:

case 1 : $\underline{\alpha < 1 \text{ or } \beta < 1}$ (Due to symmetry consider w.l.o.g. $\alpha < 1$)

Pick $x_1 = 1, x_2 = 0, x_3 = -0.5 \Rightarrow \mathbf{x}'\mathbf{C}\mathbf{x} = \alpha - 1 < 0$

Pick $x_1 = 0, x_2 = 0, x_3 = 1 \Rightarrow \mathbf{x}'\mathbf{C}\mathbf{x} = 4 > 0$

Hence C is indefinite.

case 2 : $\underline{\alpha > 1 \text{ and } \beta > 1}$

Since all squared terms are multiplied by a positive constant we know that $\mathbf{x}'\mathbf{C}\mathbf{x} \geq 0$. Furthermore $\mathbf{x}'\mathbf{C}\mathbf{x} = 0 \Leftrightarrow \mathbf{x} = 0$.

Hence C is positive definite.

case 3 : $\underline{\alpha = 1 \wedge \beta \geq 1 \text{ or } \alpha \geq 1 \wedge \beta = 1}$

Since all squared terms are multiplied by a nonnegative constant we know that $\mathbf{x}'\mathbf{C}\mathbf{x} \geq 0$

However, if $\alpha = 1$ pick $x_1 = 2, x_2 = 0, x_3 = -1$ and $\mathbf{x}'\mathbf{C}\mathbf{x} = 0$ and if $\beta = 1$ pick $x_1 = 0, x_2 = 2, x_3 = -1$ and $\mathbf{x}'\mathbf{C}\mathbf{x} = 0$

Hence C is positive semi-definite.

Problem 5

Which of the following function is homogeneous? If it is homogeneous, give the degree of homogeneity.

a) $f(x,y) = 3x^5y + 2x^2y^4 - 3x^3y^3$

b) $f(x,y) = 3x^5y + 2x^2y^4 - 3x^3y^4$

c) $f(x,y) = 3\frac{\sqrt{x}}{\sqrt{y}} + 2\frac{x}{y} + 7$

d) $f(x,y) = \frac{(x^2-y^2)}{(x^2+y^2)} + 6$

Ans:

a)
$$\begin{aligned} f(\alpha x, \alpha y) &= 3(\alpha x)^5(\alpha y) + 2(\alpha x)^2(\alpha y)^4 - 3(\alpha x)^3(\alpha y)^3 \\ &= 3\alpha^5x^5\alpha y + 2\alpha^2x^2\alpha^4y^4 - 3\alpha^3x^3\alpha^3y^3 \\ &= \alpha^6x^5y + \alpha^6 2x^2y^4 - \alpha^6 3x^3y^3 \\ &= \alpha^6 f(x,y) \end{aligned}$$

The function is homogeneous of degree 6.

b)
$$\begin{aligned} f(\alpha x, \alpha y) &= 3(\alpha x)^5(\alpha y) + 2(\alpha x)^2(\alpha y)^4 - 3(\alpha x)^3(\alpha y)^4 \\ &= 3\alpha^5x^5\alpha y + 2\alpha^2x^2\alpha^4y^4 - 3\alpha^3x^3\alpha^4y^4 \\ &= \alpha^6x^5y + \alpha^6 2x^2y^4 - \alpha^6 3x^3y^4 + \alpha^6 3x^3y^4 - \alpha^7 3x^3y^4 \\ &= \alpha^6 f(x,y) + \underbrace{\alpha^6(1-\alpha)3x^3y^4}_{\neq 0 \text{ if } \alpha=2, x=y=1} \end{aligned}$$

The function is not homogeneous.

c)
$$\begin{aligned} f(\alpha x, \alpha y) &= 3\frac{\sqrt{\alpha x}}{\sqrt{\alpha y}} + 2\frac{\alpha x}{\alpha y} + 7 \\ &= 3\frac{\sqrt{\alpha}\sqrt{x}}{\sqrt{\alpha}\sqrt{y}} + +2\frac{\alpha x}{\alpha y} + 7 \\ &= 3\frac{\sqrt{x}}{\sqrt{y}} + +2\frac{x}{y} + 7 \\ &= \alpha^0 f(x,y) \end{aligned}$$

The function is homogeneous of degree 0.

d)
$$\begin{aligned} f(\alpha x, \alpha y) &= \frac{((\alpha x)^2 - (\alpha y)^2)}{((\alpha x)^2 + (\alpha y)^2)} + 6 \\ &= \frac{(\alpha^2x^2 - \alpha^2y^2)}{(\alpha^2x^2 + \alpha^2y^2)} + 6 \\ &= \frac{\alpha^2(x^2-y^2)}{\alpha^2(x^2+y^2)} + 6 \end{aligned}$$

$$= \frac{(x^2 - y^2)}{(x^2 + y^2)} + 6$$
$$= \alpha^0 f(x, y)$$

The function is homogeneous of degree 0.

Problem 6

In the following consider functions that map from the same domain to the same range.

a) Show that the product of two homogeneous functions is homogeneous again.

b) Show that the sum of two homogenous function is homogeneous if they are homogeneous of the same degree.

c) Is it possible that two functions f and g are both homogeneous of degree k and the sum f+g is homogeneous of a degree different from k? If yes, give an example, if not, explain why.

Ans: In the following I use the definition from Smon & Blume. (Note, if you used Leo's definition that was fine too).

(i) Let f be homogeneous of degree k_1, i.e. $\forall \alpha > 0 \ f(\alpha x) = \alpha^{k_1} f(x)$

(ii) Let g be homogeneous of degree k_2, i.e. $\forall \alpha > 0 \ f(\alpha x) = \alpha^{k_2} g(x)$

a) *Show that h(x) = f(x)g(x) is a homogeneous function.*

$$\begin{aligned}\forall \alpha > 0: \ h(\alpha x) &= f(\alpha x) g(\alpha x) && \text{(by the definition of h)}\\ &= \alpha^{k_1} f(x) \alpha^{k_2} g(x) && \text{(as f,g homogeneous)}\\ &= \alpha^{k_1+k_2} f(x) g(x)\\ &= \alpha^{k_1+k_2} h(x)\end{aligned}$$

Hence h(x) is homogeneous of degree $k_1 + k_2$.

b) *Show that if $k_1 = k_2 = k$ then h(x) = f(x) + g(x) is a homogeneous function.*

$$\begin{aligned}\forall \alpha > 0: \ h(\alpha x) &= f(\alpha x) + g(\alpha x) && \text{(by the definition of h)}\\ &= \alpha^{k} f(x) + \alpha^{k} g(x) && \text{(as f,g homogeneous)}\\ &= \alpha^{k} \left(f(x) + g(x)\right)\\ &= \alpha^{k} h(x)\end{aligned}$$

Hence h(x) is homogeneous of degree k.

c) *Is it possible that if $k_1 = k_2 = k$ then h(x) = f(x) + g(x) is homogeneous of a degree different than k.*

Yes, let $g(x) = x, f(x) = -x$. Clearly f, and g are homogeneous of the same degree. Note that $\forall x: \ h(x) = 0$. This function is homogeneous of any degree, e.g, $2k + 1$.

Problem 7

a) Prove Euler's theorem (Hint: Simon & Blume).

b) Prove that if $f(x_1, x_2)$ is a twice continously differentiable function that is homogeneous of degree k then

$x_1^2 f_{x_1x_1} + 2x_1x_2f_{x_1x_2} + x_2^2 f_{x_2x_2} = k(k-1)f$

Ans: Recall Euler's theorem:

- Let $f(\mathbf{x})$ be a contiously differentiable function of degree k on $\mathbb{R}^n$ than: $x_1\frac{\delta f(\mathbf{x})}{\delta x_1} + x_2\frac{\delta f(\mathbf{x})}{\delta x_2} + ... + x_n\frac{\delta f(\mathbf{x})}{\delta x_n} = kf(\mathbf{x})$

a) *Prove Euler's theorem*
   (1) Since f is homogeneous of degree k we know that $f(\alpha\mathbf{x}) = \alpha^k f(\mathbf{x})$
   (2) Differentiate both sides with respect to α to obtain:
   $x_1\frac{\delta f(\alpha\mathbf{x})}{\delta x_1} + x_2\frac{\delta f(\alpha\mathbf{x})}{\delta x_2} + ... + x_n\frac{\delta f(\alpha\mathbf{x})}{\delta x_n} = k\alpha^{k-1}f(\mathbf{x})$
   (3) Evaluate (2) for $\alpha = 1$: $x_1\frac{\delta f(\mathbf{x})}{\delta x_1} + x_2\frac{\delta f(\mathbf{x})}{\delta x_2} + ... + x_n\frac{\delta f(\mathbf{x})}{\delta x_n} = kf(\mathbf{x})$

b) $f(x_1, x_2)$ *is a twice continously differentiable function that is homogeneous of degree k*
   (1) Since f is homogeneous of degree k we know that
   $f(\alpha x_1, \alpha x_2) = \alpha^k f(x_1, x_2)$.
   (2) Differentiate both sides with respect to α to obtain:
   $x_1\frac{\delta f(\alpha x_1,\alpha x_2)}{\delta x_1} + x_2\frac{\delta f(\alpha x_1,\alpha x_2)}{\delta x_2} = k\alpha^{k-1}f(\mathbf{x})$
   (3) Again differentiate both sides with respect to α to obtain:
   $x_1^2\frac{\delta^2 f(\alpha x_1,\alpha x_2)}{\delta x_1^2} + x_1x_2\frac{\delta^2 f(\alpha x_1,\alpha x_2)}{\delta x_2 x_1} + x_2x_1\frac{\delta^2 f(\alpha x_1,\alpha x_2)}{\delta x_1 x_2} + x_2^2\frac{\delta^2 f(\alpha x_1,\alpha x_2)}{\delta x_2^2} = k(k-1)\alpha^{k-2}f(\mathbf{x})$
   (4) Using the symmetry of the cross-partials in (3):
   $x_1^2\frac{\delta^2 f(\alpha x_1,\alpha x_2)}{\delta x_1^2} + 2x_1x_2\frac{\delta^2 f(\alpha x_1,\alpha x_2)}{\delta x_1 x_2} + x_2^2\frac{\delta^2 f(\alpha x_1,\alpha x_2)}{\delta x_2^2} = k(k-1)\alpha^{k-2}f(\mathbf{x})$
   (5) Evaluate (4) for $\alpha = 1$:
   $x_1^2\frac{\delta^2 f(x_1,x_2)}{\delta x_1^2} + 2x_1x_2\frac{\delta^2 f(x_1,x_2)}{\delta x_1 x_2} + x_2^2\frac{\delta^2 f(x_1,x_2)}{\delta x_2^2} = k(k-1)f(\mathbf{x})$