

PROBLEM SET #08

SECOND CALCULUS PROBLEM SET

DUE DATE: NOV 06

- (1) Consider the function $f(x, y, z) = xyz$, with $y = x^2$ and $z = x^{1/3}$.
- Rewrite f as a function $g : \mathbb{R} \rightarrow \mathbb{R}$ alone and compute $g'(\cdot)$. Using g' , approximate the change in f when x increases by 0.1 units, starting from $(8, 64, 2)$.
 - Compute the total derivative of f with respect to x . Using the total derivative, approximate the change in f when x increases by 0.1 units, starting from $(8, 64, 2)$.
 - Write down the differential of f at $(8, 64, 2)$. Using the differential, approximate the change in f when x increases by 0.1 units, starting from $(8, 64, 2)$.
 - Identify the direction h^* that (x, y, z) moves in, starting from $(8, 64, 2)$, when x increases. Write down the directional derivative of f in the direction \mathbf{h}^* , i.e., $f_{\mathbf{h}^*}(\cdot, \cdot, \cdot)$, and evaluate this derivative at $(8, 64, 2)$. Using $f_{\mathbf{h}^*}(8, 64, 2)$, approximate the change in f when x increases by 0.1 units, starting from $(8, 64, 2)$.
 - Check to see that all four of these distinct methods give you the same answer!
- (2) Recall that a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is nothing more than m functions, $f^1 \dots f^m$, each mapping $\mathbb{R}^n \rightarrow \mathbb{R}$, and stacked on top of each other.
- Using this fact, write down a formal definition of the directional derivative of f at x_0 in the direction $\mathbf{h} \in \mathbb{R}^n$, for a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$. Your definition should be of the form

$$\text{blah, blah} = \lim_{k \rightarrow \infty} \frac{\text{blah} \quad \text{blah}}{\text{blah, blah}}$$

(b) Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by, for $i = 1, 2$, $f^i(x, y) = x^{i/3}y^{1-i/3}$. Using the formal definition in (a) above, compute $f_{\mathbf{h}^*}(27, 8)$, where $\mathbf{h}^* = (54, 16)$. (Hint: $(27, 8) + (54, 16)/k = (27, 8)(1 + 2/k)$).

(c) Now compute $f_{\mathbf{h}^*}(27, 8)$ using the differential of f at $(27, 8)$.

(d) Check to see that all of these three distinct methods give you the same answer!

(3) Consider the function $f(\mathbf{x}) = x_1^\rho + x_2^\rho$, where $\rho \in (-\infty, 1]$. *The whole point here is to use the differential of ∇f to answer the following questions, i.e., to answer all parts of the question, approximate $\nabla f(\mathbf{x} + \mathbf{h}) - \nabla f(\mathbf{x})$ using the differential of ∇f at \mathbf{x} , evaluated at \mathbf{h} . There are lots of other ways to answer these questions, but the purpose of this question is to give you practice in using the differential of a vector-valued function.*

(a) Check that, up to a first order approximation,¹ f is homothetic (cf the notes for lecture CALCULUS3,² specifically the second example in the subsection entitled Four Graphical Examples.³)

(b) When $\rho > 0$, does f exhibit increasing, constant or decreasing returns to scale? Is your answer true for all $\rho \in (0, 1]$. (Again, your answer should be in terms of what happens to the *gradient vector* as you move out along a ray.)

(c) Fix $\mathbf{x} = (\alpha, \alpha)$, and consider $\mathbf{h} = (-0.1, 0.1)$. Approximate $\nabla f(\mathbf{x} + \mathbf{h})$, for (i) $\rho = 1/2$; (ii) $\rho = -1/2$; (iii) $\rho = -10$.

(d) How does the curvature of the level sets of this function change as you move out along a ray through the origin. In particular, discuss the effect of the magnitude of α on the *rate of change* in the direction of ∇f as you add $\mathbf{h} = (-\beta, \beta)$ to $\mathbf{x} = (\alpha, \alpha)$.

¹ The qualifier “up to a first order approximation” means: you should *pretend* that the answer you get using the differential is exactly correct, even though in fact it is only approximately correct, and then only for small \mathbf{h} 's, because there are non-zero higher order terms in the Taylor expansion of ∇f .

² In the example in the notes, you don't need the caveat about up to a first order approximation, because the higher order terms in the Taylor approximation are all zero. In this example they are not.

³ The lecture notes tend to change, and sometimes the problem sets don't keep up. If this reference is no longer current, please notify Leo.

- (4) Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is $(n+1)$ times continuously differentiable.
- Show that a sufficient condition for f to attain a strict (local) maximum at x_0 is that for some even number n , the derivatives $f^{(k)}(x_0)$ are zero for $k = 1 \dots n - 1$, and $f^{(n)}(x_0)$ is negative.
 - If $f^{(k)}(x_0)$ is zero for $k = 1 \dots n - 1$ and $f^{(n)}(x_0)$ is non-zero, show that there exists an ϵ -neighborhood around x_0 where the absolute value of the n^{th} -order Taylor expansion is bigger than the absolute value of the remainder term $R_n(x)$.
 - Give a counter example to show that the result in part (a) would be false if the words “for some even n ” were replaced with “for some $n > 0$ ”.
 - Explain carefully, but in as few words as possible, why the argument in (a) works for even n but not for odd n .
 - Show that the n^{th} -order Taylor expansion around any point x_0 of a polynomial of degree n (i.e. a function of the form $f(x) = \sum_{k=0}^n \alpha_k x^k$) is perfectly accurate, regardless of the magnitude of dx .
 - Show that if f is an arbitrary polynomial of degree 2, i.e., $f(x) = ax^2 + bx + c$, then for any point x_0 , if you add to $f(x_0)$ the 2^{nd} -order Taylor expansion around f , the expression you get is precisely the original function f . More precisely, show by writing out the Taylor expansion explicitly, that for arbitrary dx ,

$$f(x_0 + dx) = f(x_0) + T_2(f, x_0, dx).$$