

PROBLEM SET #07- ANSWER KEY

FIRST CALCULUS PROBLEM SET

Please note: There are parts of one question in this assignment to which you may want to reply: “I beg your pardon?” or something possibly ruder than this. That kind of response is entirely appropriate, but you should add a sentence explaining why you are led to make such a remark.

(1) Consider the function $f(x) = 20x^3 - 120x^2 - 5x + 36$

a) What is the derivative of this function?

$$\text{Ans: } f'(x) = 60 * x^2 - 240 * x - 5$$

b) What is the derivative of this function evaluated at $x_0 = 4$?

$$\text{Ans: } f'(x_0 = 4) = 60 * 4^2 - 240 * 4 - 5 = 240 * 4 - 240 * 4 - 5 = -5$$

c) What is the differential of this function at $x_0 = 4$?

$$\text{Ans: } df = -5dx$$

d) Approximate the change in the function when moving from $x_0 = 4$ to $x_1 = 5$.

$$\text{Ans: } df|_{x_0=4} = -5 * 1 = -5$$

e) What is the actual change in the function when moving from $x_0 = 4$ to $x_1 = 5$?

$$\text{Ans: } \Delta f = f(x_1 = 5) - f(x_0 = 4) = -489 - (-624) = 135.$$

Not a very good approximation, huh! Any ideas why? Try plotting the function.

(2) Multivariate calculus drill: Simon and Blume, question 14.1, parts (a), (d) and (f);

Ans:

$$\text{a) } \frac{\partial f}{\partial x} = 8xy - 3y^3 + 6, \frac{\partial f}{\partial y} = 4x^2 - 9xy^2$$

$$\begin{aligned} \text{d) } \frac{\partial f}{\partial x} &= 2e^{2x+3y}, \frac{\partial f}{\partial y} = 3e^{2x+3y}. \\ \text{f) } \frac{\partial f}{\partial x} &= 6xy - 7\sqrt{y}, \frac{\partial f}{\partial y} = 3x^2 - \frac{7x}{2\sqrt{y}} \end{aligned}$$

(3) Gradients: Simon and Blume, question 14.19

Ans: $\Delta y^2 e^{3x} = (3y^2 e^{3x}, 2ye^{3x})$. At $(0, 3)$, this vector is proportional to $(9, 2)$. Normalizing to length 1, the vector is $(9/\sqrt{85}, 2/\sqrt{85})$.

(4) More differential approximations: Simon and Blume, question 14.4.

Ans:

- a) $Q = 5400$
- b) $Q(998, 216) = 5398.798$. The approximation gives $Q \approx 5392.8$, which is in error by -0.002 .
- c) $Q(1000, 217.5) = 5412.471$. The approximation gives $Q \approx 5412.5$, which is in error by -0.029 .
- d) For the approximation to be in error by more than 2.0, ΔL must be at least 58.475, or about 5.875% of L .

(5) Still more differential approximations: Simon and Blume, question 14.8b.

Ans: $df = \frac{4\sqrt{y}}{x^{1/3}} dx + \frac{3x^{2/3}}{\sqrt{y}} dy$ Evaluating at $x = 1000, dx = -2, y = 100$, and $dy = 1.5$, $f = 6037$.

(6) And one more: Simon and Blume, question 14.10

Ans: $df = \frac{3x^2}{2\sqrt{x^3-y^3-z^3}} dx - \frac{3y^2}{2\sqrt{x^3-y^3-z^3}} dy - \frac{3z^2}{2\sqrt{x^3-y^3-z^3}} dz$. Evaluating at $x = 4, dx = 0.1, y = 0.2, dy = -0.05, z = 1, \text{ and } dz = 0.02$ gives $f \approx 6.5075$.

(7) Consider the function $f(\mathbf{x}) = x_1x_2$. Let $\mathbf{x}_0 = (3, 2)$ and $\mathbf{h} = (3, 4)$.

- (a) Directly compute the directional derivative of f at \mathbf{x}_0 in the direction \mathbf{h} by the method used in class and in the lecture notes.

Ans:

$$f_{\mathbf{h}}(\mathbf{x}) = \lim_{|k| \rightarrow \infty} \frac{(f(\mathbf{x}_0 + \mathbf{h}/k) - f(\mathbf{x}_0))}{\|\mathbf{h}\|/k}$$

In this case:

$$\begin{aligned} f_{(3,4)}(1, 2) &= \lim_{|k| \rightarrow \infty} \frac{(f((3, 2) + (3, 4)/k) - f((3, 2)))}{\|(3, 4)\|/k} = \lim_{|k| \rightarrow \infty} \frac{((3 + 3/k) \times (2 + 4/k) - 6)}{5/k} \\ &= \lim_{|k| \rightarrow \infty} \frac{((6 + 18/k + 12/k^2) - 6)}{5/k} = \lim_{|k| \rightarrow \infty} \frac{18 + 12/k}{5} = \frac{18}{5} \end{aligned}$$

- (b) Indirectly compute the directional derivative of f at \mathbf{x}_0 in the direction \mathbf{h} using the differential

Ans: We evaluate the differential at the *unit length* vector pointing in the same direction as \mathbf{h} , i.e., $(3/5, 4/5)$. We now have:

$$\nabla f(\mathbf{x}) \cdot \mathbf{h} = (2, 3) \cdot (3/5, 4/5) = 6/5 + 12/5 = \frac{18}{5}$$

- (c) Do both computations give the same answer. If not, why not?

Ans: If you didn't get the right answer, you probably didn't remember to shrink \mathbf{h} so that it is a *unit length* vector.

(8) Consider the function $f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

Ans: For the record, the graph of this function is:

Note that the graph is flat in the directions parallel to the axes. Also note that the differential

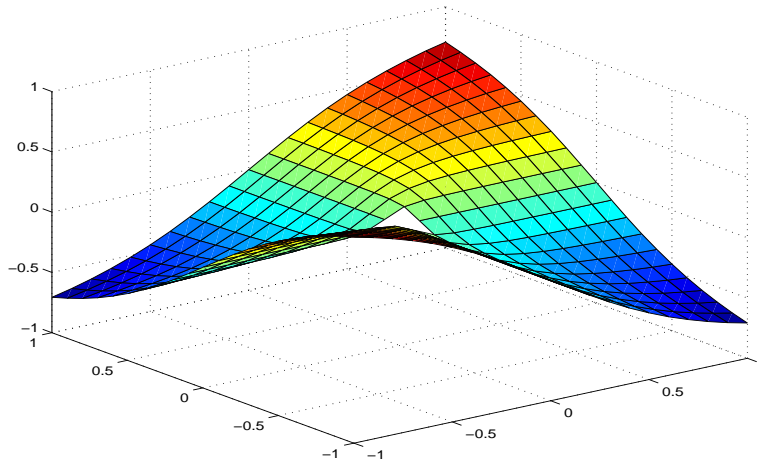


FIGURE 1. Question 8

clearly isn't going to give you the right answer!

(a) Using the chain rule, compute $f_x(x, y)$, for $(x, y) \neq (0, 0)$.

Ans:

$$f_x(x, y) = \frac{y}{\sqrt{x^2 + y^2}} - \frac{x^2 y}{(x^2 + y^2)^{3/2}}$$

(b) Compute $f_x(0, 0)$, using the following definition: $f_x(x, y) = \lim_{t \rightarrow 0} \frac{f((x, y) + (t, 0)) - f(x, y)}{t}$.

Ans:

$$\frac{f((0, 0) + (t, 0)) - f(0, 0)}{t} = \frac{\frac{0 \times t}{\sqrt{t^2}} - 0}{t} = 0$$

- (c) Compute the directional derivative of f at $(0, 0)$ in the direction $(3, 3)$, using the method presented in class (and the lecture notes).

Ans: The definition of the directional derivative of function f at $(0, 0)$ in the direction $(3, 3)$ is:

$$\lim_{|k| \rightarrow \infty} \frac{f((0, 0) + (\frac{3}{k}, \frac{3}{k})) - f(0, 0)}{\frac{\|(3, 3)\|}{k}},$$

if this limit exists. This means that if the above limit does not exist, then the directional derivative is not defined. We have:

$$f((0, 0) + (\frac{3}{k}, \frac{3}{k})) = f(\frac{3}{k}, \frac{3}{k}) = \frac{(\frac{3}{k})^2}{\sqrt{(\frac{3}{k})^2 + (\frac{3}{k})^2}} = \frac{\frac{9}{k^2}}{\frac{3\sqrt{2}}{|k|}} = \begin{cases} \frac{3}{\sqrt{2}k} & \text{if } k > 0 \\ -\frac{3}{\sqrt{2}k} & \text{if } k < 0 \end{cases},$$

so that

$$\frac{f((0, 0) + (\frac{3}{k}, \frac{3}{k})) - f(0, 0)}{\frac{\|(3, 3)\|}{k}} = \begin{cases} \frac{\frac{3}{\sqrt{2}k}}{\frac{3}{k}} = \frac{1}{\sqrt{2}} & \text{if } k > 0 \\ -\frac{\frac{3}{\sqrt{2}k}}{\frac{3}{k}} = -\frac{1}{\sqrt{2}} & \text{if } k < 0 \end{cases}.$$

Since these two numbers are not equal, we can say that $\frac{f((0, 0) + (\frac{3}{k}, \frac{3}{k})) - f(0, 0)}{\frac{\|(3, 3)\|}{k}}$ does not admit a limit when $|k| \rightarrow \infty$. Therefore, the directional derivative of f at $(0, 0)$ in the direction $(3, 3)$ is not defined. Since existence of all directional derivatives at $(0, 0)$ is a necessary condition for f to be differentiable at $(0, 0)$, we can conclude that f is not differentiable at $(0, 0)$.

This result provides you with the key to answering question (e). (Since the directional derivative does not exist, we cannot compare it to the differential of f at $(0, 0)$ evaluated at the unit length vector pointing in the direction $(3, 3)$.)

- (d) Write down $f_y(0, 0)$ and evaluate the expression $\nabla f(0, 0) \cdot (3, 3)$.

Ans: By symmetry $f_y(0, 0) = 0$, so that $\nabla f(0, 0) \cdot (3, 3) = 0$.

- (e) For $x_0 = y_0 = 0$ and $\mathbf{h} = (3, 3)$, what can be said about the expression

$$\lim_{|k| \rightarrow \infty} \frac{(f((x_0, y_0) + \mathbf{h}/k) - f(x_0, y_0)) - \nabla f(x_0, y_0) \cdot \mathbf{h}/k}{\|\mathbf{h}\|/k}?$$

Ans: The limit doesn't exist. See the answer to part (c).

- (f) What can you conclude about the differentiability or otherwise of f ?

Ans: The function isn't differentiable. If it were, the expression computed above would have been zero.

(g) With x on the horizontal axis, plot $f_x(x, x)$, for $x \in [-1, 1]$.

Ans: From the above, when $x > 0$,

$$\begin{aligned} f_x(x, x) &= \frac{x}{\sqrt{x^2 + x^2}} - 0.5 \frac{x^3}{(x^2 + x^2)^{3/2}} \\ &= \frac{1}{\sqrt{2}} - \frac{1}{2\sqrt{2}} \\ &= \frac{1}{2\sqrt{2}} \end{aligned}$$

So the graph is $f_x(x, x) = \begin{cases} \frac{3}{4\sqrt{2}} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ \frac{-3}{4\sqrt{2}} & \text{if } x < 0 \end{cases}$

(i) From your graph, what can you say about the continuity or otherwise of $f_x(\cdot, \cdot)$ at $(0, 0)$.

Ans: f_x is discontinuous at $(0, 0)$.

(ii) From your graph, what can you say about the continuity or otherwise of $f_x(0, 0)$.

Ans: Nothing, it doesn't make sense to talk about continuity of a number.

(iii) Explain why your answers to 8(g)i and 8(g)ii are different.

Ans: $f_x(\cdot, \cdot)$ is a *function* and it makes sense to talk about continuity of a function, evaluated at a point. $f_x(0, 0)$ is a *number* and, as said above, it doesn't make sense to talk about continuity of a number.

(iv) What is the relationship between your answers to 8(g)i and 8(g)ii and your answer to 8f?

Ans: No relationship to 8(g)ii, which was a silly question. The relationship between 8(g)i and 8f is simply that they are consistent with each other. It is *not true* that if

the answer to $\partial(g)_i$ had been different—if f_x had been continuous—then the answer to ∂f would have had to have been different. It is also not the case that had the answer to ∂f been different—i.e., had f been differentiable—then the answer to $\partial(g)_i$ would have had to be different also. A function can be differentiable while the partial's are not. An example is provided in the notes.