Problem Set 4 Solutions

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1 (a)

Proof using the *sequential formulation of upper hemi-continuity.*

Assume a Euclidean metric. We need to show that $\psi(C)$ is closed and bounded.

Closedness

BWOC suppose $\psi(C)$ is not closed. Then $\psi(C)$ does not contain all of it's accumulation points; let one of these accumulation points be y_0. We may construct a sequence $y_n \in \psi(C)$ s.t. $y_n \to y_0$. Note that it can be easily verified as a lemma that if $y_n \to y_0$ then any convergent subsequence of y_n must also converge to y_0.

We are given that ψ is UHC so that according to the sequential formulation of UHC then $\forall \overline{x} \in X$ every $x_n \to \overline{x}$ and $y_n \in \psi(x_n)$ then there exists a subsequence of y_n, call it y_{n_k}, such that $y_{n_k} \to \overline{y} \in \psi(\overline{x}) \subset \psi(C)$. By the lemma above we know $y_0 = \overline{y} \in \psi(C)$, and we have arrived at a contradiction. Hence, $\psi(C)$ is closed.

Boundedness

BWOC suppose $\psi(C)$ isn't bounded so that $\forall n \in \mathbb{N}$, $\exists x \in X$ such that $\max\{\psi(x)\} > n$. We begin by constructing a sequence x_n and a corresponding sequence y_n. However, before doing so we note that since ψ is compact-valued we know that $\forall x$ that the max $\{\psi(x)\} \in \psi(x)$. We now construct our sequence in the following way:

- Choose $x_1 = x$ such that $\max\{\psi(x)\} > 1$, and let $y_1 = \max\{\psi(x_1)\}$. By the Archimedian Property we know there exists $N_1 \in \mathbb{N}$ such that $N_1 = 1 + \ulcorner y_1 \urcorner$;
- Choose $x_2 = x$ such that $\max\{\psi(x)\} > N_1$, and let $y_2 = \max\{\psi(x_2)\}$. By the Archimedian Property we know there exists $N_2 \in \mathbb{N}$ such that $N_2 = 1 + \ulcorner y_2 \urcorner$;
- Choose $x_3 = x$ such that $\max\{\psi(x)\} > N_3$, and let $y_3 = \max\{\psi(x_3)\}$;

and so on we construct x_n and y_n. Note that this implies $\forall m, n \in \mathbb{N}\,(n \neq m)$ (WLOG let $m > n$) we have:

$$d(y_m, y_n) = y_m - y_n \geq y_{n+1} - y_n > N_n - y_n = 1 + \ulcorner y_n \urcorner - y_n > 1 \tag{1}$$

Moving forward, we are given X is compact (and so is bounded) so we know by the Bolzano-Weierstrass Theorem that x_n contains a convergent subsequence x_{n_k}, with which we can associate the sequence $\{y_{n_k}\} \in \psi(C)$. Let $x_{n_k} \to \overline{x} \in X$ as $n \to \infty$.

Now according to the sequential formulation of UHC, we know for $x_{n_k} \to \overline{x} \in X$ and $y_{n_k} \in \psi(x_{n_k})$ that there is a convergent subsequence of $\{y_{n_k}\}$, call it $\{y_{n_{k_j}}\}$ such that $y_{n_{k_j}} \to \overline{y} \in \psi(\overline{x})$ as $n \to \infty$. Hence, for $\epsilon = \frac{1}{2}$, $\exists N \in \mathbb{N}$ such that $\forall n$, $m > N$ we have,

$$d(y_m, \overline{y}) < \epsilon \quad \text{and} \quad d(y_n, \overline{y}) < \epsilon$$

By the triangle inequality we know,

$$d(y_m, y_n) \leq d(y_m, \overline{y}) + d(y_n, \overline{y}) < 2\epsilon = 1$$

We have arrived at a contradiction and so our supposition is false; $\psi(C)$ is bounded.

$\therefore$ $\psi(C)$ is closed and bounded, and so is compact.

1(b)

Proof using alternative formulations of upper hemi-continuity.

Since we are using the closed-bounded defn of compactness, it had better be the case that X and Y are Euclidean spaces.

Closedness

Suppose $\Psi(C)$ is not closed, i.e., there's a sequence $\{x_n, y_n\} \in \Psi(C)$ such that $\{x_n, y_n\} \to (x, y)$ but the limit $(x, y) \notin \Psi(C)$, i.e., $y \notin \Psi(x)$. Since $\Psi(x)$ is compact, y is not a boundary point of $\Psi(x)$, i.e., there exists an open set U containing y such that $U \cap \Psi(x) = \emptyset$. (I think the existence of U is obvious but perhaps technically we should prove it?) Let O be an open set containing $\Psi(x)$ such that $U \cap O = \emptyset$. Now consider any nbd V of x. Since $\{x_n\}$ converges to x, there exists N such that for $n > N$, $x_n \in V$. Since y_n converges to y, there exists $N' > N$ such that for $n > N'$, $y_n \in U$. But $y_n \in \Psi(x_n)$ and $U \cap O = \emptyset$, so that $\Psi(x_n) \nsubseteq O$. Thus Ψ fails the $\epsilon - \delta$ defn of u.h.c. This argument also proves that x is a boundary point of $\bar{\Psi}^{-1}(O)$, showing that Ψ fails the third defn of u.h.c.

Boundedness

Suppose $\Psi(C)$ is not bounded above, i.e., there's a sequence $\{x_n\} \in C$ and an element $y_n \in \Psi(x_n)$ such that for all n, $||y_n|| > n$. Since C is compact, there's a subsequence of $\{x_n\}$ which converges to $x \in X$. Let's just assume that $\{x_n\}$ itself converges to x. $\Psi(x)$ is compact and hence bounded. Let O be a bounded open set containing $\Psi(x)$, let b be a bound for O and let V be any nbd of x. There exists N such that for $n > N$, $x_n \in V$. For $N' > N > b + 1$, $||y_n|| > b$ so that $\Psi(x_n) \nsubseteq O$. Thus Ψ fails the $\epsilon - \delta$ defn of u.h.c. This argument also proves that x is a boundary point $\bar{\Psi}^{-1}(O)$, showing that Ψ fails the third defn of u.h.c.

$\therefore$ $\psi(C)$ is closed and bounded, and so is compact.

2(a)

We want to show that $\underline{\Psi_\epsilon}^{-1}(O)$ not open implies that $O \subset \mathbb{R}^m$ is not open. Let $\underline{\Psi_\epsilon}^{-1}(O)$ be not open so that $\exists w \in \underline{\Psi_\epsilon}^{-1}(O)$, call it w_0, such that $\forall \epsilon > 0$, $B_\epsilon(w_0)$ contains infinitely many $w \notin \underline{\Psi_\epsilon}^{-1}(O)$. Now we may construct $\{w_n\}_{n=1}^{\infty}$ converging to w_0 such that $\forall n$, $w_n \notin \underline{\Psi_\epsilon}^{-1}(O)$. Also, note that $\exists x_0 \in \Psi_{\epsilon-\delta}(w_0) \subset \Psi_\epsilon(w_0)$.

Now pick an $\{x_n\}$ converging to x_0. We are given F is continuous so that $(x_n,w_n) \to (x_0, w_0)$ implies $F(x_n, w_n) - F(x_0, w_0) < \delta$ for all $n > N$ for some $N \in \mathbb{N}$; as $x_0 \in \Psi_{\epsilon-\delta}(w_0)$ we have $F(x_n, w_n) < \epsilon$. Hence, $x_n \in \Psi_\epsilon(w_n)$, and as $w_n \notin \underline{\Psi_\epsilon}^{-1}(O)$ it must be that $x_n \notin O$. We have constructed a sequence $x_n$converging to x_0 where $x_n \notin O$ for all n. It follows that x_0 is a boundary point of O and so O is not open.

$\therefore \Psi_\epsilon$ is l.h.c. for all $\epsilon > 0$.

2(b)

Consider the sequence (x_n, w_n) such that $w_n \to w_0$ as $n \to \infty$, and $x_n \in \Psi_\epsilon(w_n)$. It follows that $F(x_n,w_n) = 0$ for all n . Now note that since x_n lives in a compact set we know there exists a subsequence x_{n_k} that converges to some x_0; and so we have $(x_n, w_n) \to (x_0, w_0)$. Hence, by the contintuity of F we know that $F(x_n, w_n) \to F(x_0, w_0)$. Since $\{F(x_n, w_n) = 0\}_{n=1}^{\infty}$ and

$F(x_n, w_n)$ converges, then $F(x_0, w_0) = 0$.

$\therefore$ $\Psi(w)$ is u.h.c..

2(c)

Let $F(x, \omega) = \frac{1}{x} - \omega$ where $C = \mathbb{R}/0$ which is not compact. Thus F is clearly continuous on $C \times \mathbb{R}$ defined on $\mathbb{R}/0$. Now,

$$\Psi(\omega) = \begin{cases} \frac{1}{\omega} & \text{if } \omega \neq 0 \\ \emptyset & \text{if } \omega = 0 \end{cases}$$

$\Psi(\omega)$ is not U.H.C..

3(a)

Follows from the Extreme Value Theorem.

3(b)

Consider the sequence $\{(x_n, w_n)\}$ such that $w_n \to w_0$ as $n \to \infty$, and $x_n \in \Psi(w_n)$. We know that max $F(x_n, w_n) = \sup F(x_n, w_n)$ for all n . Now note that since x_n lives in a compact set we know there exists a subsequence x_{n_k} that converges to some x_0; and so we have $(x_{n_k}, w_{n_k}) \to (x_0, w_0)$. By the contintuity of F we know that $F(x_{n_k}, w_{n_k}) \to F(x_0, w_0)$. And we know for any $\frac{\epsilon}{2}$-neighborhood of $F(x_0, w_0)$, $B_{\frac{\epsilon}{2}}(F(x_0, w_0))$, we know there exists a δ-neighborhood, $V_\delta(w_0)$ such that $F(x_0, w) \in B_{\frac{\epsilon}{2}}(F(x_0, w_0))$ for all $w_{n_k} \in V_\delta(w_0)$. This implies:

$$F(x_0, w_{n_k}) < F(x_0, w_0) + \frac{\epsilon}{2} \tag{2}$$

for all $w_{n_k} \in V_\delta(w_0)$.

An alternative way to think of this is to note that since $F(x_{n_k}, w_{n_k}) \to F(x_0, w_0)$ as $n \to \infty$ we can pick any $\frac{\epsilon}{2} > 0$ and any δ and find $N \in \mathbb{N}$ such that $\forall n > N$:

$$|F(x_{n_k}, w_{n_k}) - F(x_0, w_0)| < \frac{\epsilon}{2} \quad \to \quad F(x_{n_k}, w_{n_k}) < F(x_0, w_0) + \frac{\epsilon}{2}$$

Now we only need to show that $x_0 \in \Psi(w_0)$, that is $F(x_0, w_0) = \sup F(z, w_0)$ for $z \in X$. Suppose not, then $\exists x'$ such that $F(x', w_0) > F(x_0, w_0)$. Let $\epsilon = F(x', w_0) - F(x_0, w_0)$.

Again considering the continuity of F, we know for any $\frac{\epsilon}{2}$-neighborhood of $F(x', w_0)$, $B_{\frac{\epsilon}{2}}(F(x', w_0))$, we know there exists a δ-neighborhood, $V_\delta(w_0)$ such that $F(x', w) \in B_{\frac{\epsilon}{2}}(F(x', w_0))$ for all $w \in V_\delta(w_0)$. This implies:

$$F(x', w_0) - \frac{\epsilon}{2} < F(x', w) \tag{3}$$

for all $w \in V_\delta(w_0)$.

Let $w = w_{n_k} \in V_\delta(w_0)$, so that by (1) and (2) we have:

$$F(x_{n_k}, w_{n_k}) < F(x_0, w_0) + \frac{\epsilon}{2} = F(x', w_0) - \frac{\epsilon}{2} < F(x', w_{n_k}) \qquad (4)$$

And we find that for $x = x'$, then $F(x_{n_k}, w_{n_k}) < F(x', w_{n_k})$ and $x_{n_k} \notin \Psi(w_{n_k})$. We have arrived at a contradiction and $x_0 \in \Psi(w_0)$.

$\therefore \Psi$is u.h.c..

3(c)

We adapt the example from 2(c). Let $F(x, \omega) = -(\frac{1}{x} - \omega)^2$ where $C = \mathbb{R}/0$ which is not compact. Thus F is clearly continuous on $C \times \mathbb{R}$ defined on $\mathbb{R}/0$. Now,

$$\Psi(\omega) = \begin{cases} \frac{1}{\omega} & \text{if } \omega \neq 0 \\ \emptyset & \text{if } \omega = 0 \end{cases}$$

$\Psi(\omega)$ is not U.H.C..