

PROBLEM SET #03

THIRD ANALYSIS PROBLEM SET

DUE DATE: SEP 25

Problem 1

For the following problem, consider an arbitrary universe X and an arbitrary metric d defined on $X \times X$. State whether the following statements are true or false. If they are true, give a proof. If they are wrong, give a counter-example

- a) If a sequence x_n converges then every subsequence of x_n must also converge.
- b) If *every* subsequence of the sequence x_n converges, then x_n must also converge.
- c) If *every* subsequence of the sequence x_n converges, then they all converge to the same point.
- d) If *one* subsequence of the sequence x_n converges, then x_n must also converge.

Problem 2

For the following problem, consider an arbitrary universe X and an arbitrary metric d defined on $X \times X$. You are given a sequence x_n . Consider the set $S = \{x_n\}_{n=1}^{\infty}$ (The set consists of all elements in the sequence). Prove that if b is an accumulation point of the set S , then some subsequence of x_n converges to b .

Problem 3

Let (x_n) and (y_n) be sequences in \mathbb{R}^m such that both $\lim_n x_n$ and $\lim_n (x_n \cdot y_n)$ exist.

- (1) When $m = 1$, prove that $\lim_n (y_n)$ need not exist.
- (2) When $m = 1$, prove that if $\lim_n x_n \neq 0$ then $\lim_n (y_n)$ exists.

- (3) When $m = 2$, does $\lim_n x_n \neq 0$ imply that $\lim_n (y_n)$ necessarily exists. If so, prove it. If not, provide a counter-example.

Problem 4

Consider the following function $\rho : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, defined by

$$\rho(x, y) = \begin{cases} \left| \frac{1}{x} - \frac{1}{y} \right| & \text{if } x, y > 0 \\ 0 & \text{if } x = y = 0 \\ 1/x & \text{if } y = 0 \\ 1/y & \text{if } x = 0 \end{cases}$$

- (1) Verify that ρ is a metric. (Checking the triangle inequality is a bit fiddly. You should first consider the case of $x, y > 0$. In my answer, I had to consider three separate cases; hopefully some of you will figure out something more elegant. Then suppose that either x or y is zero. I had to consider two cases here.)
- (2) Identify (and verify) a necessary and sufficient condition for the set $I^\alpha = [\alpha, \infty) \subset \mathbb{R}$, $\alpha \geq 0$, to be closed in \mathbb{R}_+ under the metric ρ . If I^α is not closed for some $\alpha \geq 0$, write down the closure of I^α in \mathbb{R}_+ under the metric ρ .
- (3) Identify and verify a necessary and sufficient condition for the set $I^\alpha = [\alpha, \infty) \subset \mathbb{R}$, $\alpha \geq 0$, to be bounded under the metric ρ and the corresponding norm.
- (4) Consider the following two functions. (In each case, the metric on the range is Pythagorean.)
 - (a) $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, defined by $f(x) = \begin{cases} 1/x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$. Prove that this function is continuous at zero when ρ is the metric on the domain. (The metric on the range is Pythagorean.)

(b) $f : Z_+ \rightarrow \mathbb{R}_+$, defined by $f(z) = 1/(z+1)$, where Z_+ denotes the nonnegative integers, i.e., $0, 1, 2, \dots$

(i) Is this function continuous when ρ is the metric on the domain?

(ii) What if the Pythagorean metric is the metric on the domain?

(5) When ρ is the metric on \mathbb{R}_+ , zero is not the most natural word to apply to the symbol “0”. A more natural word would be _____. Fill in the blank with one word: there is exactly one correct word. In *one sentence or less*, explain why this word is more natural.