# Problem Set #03- Answer key

## Third Analysis Problem Set

### Problem 1

For the following problem, consider an arbitrary universe X and an arbitrary metric d defined on $X \times X$. State whether the following statements are true or false. If they are true, give a proof. If they are wrong, give a counter-example

a) If a sequence $x_n$ converges then every subsequence of $x_n$ must also converge.

b) If *every* subsequence of the sequence $x_n$ converges, then $x_n$ must also converge.

c) If *every* subsequence of the sequence $x_n$ converges, then they all converge to the same point.

d) If *one* subsequence of the sequence $x_n$ converges, then $x_n$ must also converge.

**Ans:** Recall the definitions

(C) *Convergence*: A sequence $x_n$ in X converges to an element $\bar{x} \in X$ in the metric d if $\forall\ \epsilon > 0\ \exists N \in \mathbb{N}$ such that $\forall n > N$ we have $d(x_n, \bar{x}) < \epsilon$

(S) *Subsequence*: A sequence $\{y_1, y_2, ..., y_n, ...\}$ is a subsequence of another sequence if there exists a strictly increasing function $\tau : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N} : y_n = x_{\tau(n)}$

Let's first prove the following lemma (L) by induction: $\forall$ strictly increasing $\tau : \mathbb{N} \to \mathbb{N}$ we must have $\tau(n) \geq n\ \forall n \in \mathbb{N}$

(i) *initialization:* for n = 1: $\tau(n) \geq 1$ as $\tau$ maps from $\mathbb{N}$ into $\mathbb{N}$.

(ii) *induction hypothesis:* $\forall n \leq \bar{n} : \tau(n) \geq n$.

(iii) *induction step:* $\bar{n} \to \bar{n} + 1$

$$\begin{aligned}\tau(\bar{n}+1) &> \tau(\bar{n}) && \text{(as } \tau() \text{ is strictly increasing)}\\ &\geq \bar{n} && \text{(From (ii)).}\end{aligned}$$

Since $\tau(\bar{n}+1) \in \mathbb{N}$ we therefore know that $\tau(\bar{n}+1) \geq \bar{n}+1$.

Let's consider the problems:

a) True: If a sequence $x_n$ converges then so does every subsequence of $x_n$.
   (1) You are given that a sequence $x_n$ converges, i.e., $\exists \bar{x} \in X$ s.t. $\forall\ \epsilon > 0\ \exists N \in \mathbb{N}$ such that $\forall n > N$ we have $d(x_n, \bar{x}) < \epsilon$
   (2) Using lemma (L) in the definition of a subsequence (S) we know that $\forall n : y_n = x_{\tau(n)} = x_{\hat{n}}$ for some $\hat{n} \geq n$.
   (3) But (2) and (1) imply that $\forall\ \epsilon > 0\ \exists N \in \mathbb{N}$ such that $\forall n > N$ we have $d(y_n, \bar{x}) < \epsilon$. Hence $y_n$ converges to $\bar{x}$.

b) True: If *every* subsequence of the sequence $x_n$ converges then so does $x_n$. This is trivial: Since *every* subsequence convergence, pick the subsequence $\tau(n) = n$ which is simply the sequence itself. Hence the sequence converges.

c) True: from b), $(x_n)$ itself converges. Let $\bar{x}$ be the point to which it converges. Fix $\epsilon > 0$; $\exists N^\epsilon \in \mathbb{N}$ such that $\forall n > N^\epsilon$, $d(x_n, x) < \epsilon$. Now consider an arbitrary subsequence $(y_n)$ of $(x_n)$ and let $\tau(\cdot)$ denote the strictly increasing function from $\mathbb{N}$ to $\mathbb{N}$ that defines $(y_n)$. From class we know that $\tau(\cdot)$ increases without bound. Hence there exists $M \in \mathbb{N}$, such that for all $m > M$, $\tau(m) > N^\epsilon$ so that $y_m = x_{\tau(m)} \in B(\bar{x}, \epsilon)$. Hence $(y_n)$ converges to $\bar{x}$.

## Problem 2

For the following problem, consider an arbitrary universe X and an arbitrary metric d defined on $X \times X$. You are given a sequence $x_n$. Consider the set $S = \{x_n\}_{n=1}^{\infty}$ (The set consists of all elements in the sequence). Prove that if b is an accumulation point of the set S, then some subsequence of $x_n$ converges to b.

**Ans:** Recall the folowing definitions:

(C) *Convergence*: A sequence $x_n$ in X converges to an element $b \in X$ in the metric d if $\forall\ \epsilon > 0\ \exists N \in \mathbb{N}$ such that $\forall n > N$ we have $d(x_n, b) < \epsilon$

(A) *Accumulation Point*: A point $b \in X$ is called an accumulation point of a set $S \subset X$ if $\forall \epsilon > 0$ the ball $B_d(b, \epsilon)$ contains a point $x \in S, x \neq b$.

As a first step to this problem we will show that every $\epsilon$-ball around an accumulation point includes infinitely many points. Proof by contradiction:

(1) Assume the $\epsilon$-ball contains only finitly many points $x_1, ...., x_N$ for some $N \in \mathbb{N}$

(2) Define $\delta = min\{d(x_n, b) | n = 1...N, x_n \neq b\} > 0$, i.e., $\delta$ is the smallest distance between the accumulation point b and any of the finitely many elements in the ball that do not equal b.

(3) Note that by the definition of $\delta$ in (2) and the fact that all $x_n$ are an element of the $\epsilon$-ball around b, $\delta < \epsilon$

(4) Hence the $\epsilon$-ball $B_d(b, \delta)$ contains no points other than b. (If it would contain a point $x_{\bar{n}}$ it would lie in the $\epsilon$-ball of (1) as $\delta < \epsilon$ and hence would contradict the definition of $\delta$ in (2) as $d(x_{\bar{n}}, b) < \delta$).

(5) Thus, (4) is a contradiction to the fact that b is an accumulation point.

Now let's prove the question by construction:

(1) Let $\tau(1) = 1$ and $\forall n \in \mathbb{N}$ with $n > 1$ let $\tau(n) = min\{n \in \mathbb{N} \mid n > \tau(n-1), d(x_n, b) < \frac{1}{n}\}$. Since every ball around an accumulation point b includes infinitely many points we know that we can construct such a $\tau(n)$.

(2) From the construction we know that $\tau(n)$ is a strictling increasing function that maps from $\mathbb{N}$ to $\mathbb{N}$

(3) From (2) we know that $y_n = x_{\tau(n)}$ is a subsequence of $x_n$ and from construction in (1) we know that $\forall\ \epsilon > 0\ \exists N = \lceil \frac{1}{\epsilon} \rceil \in \mathbb{N}$ such that $\forall n > N$ we have $d(y_n, b) < \epsilon$. Hence $y_n$ is a subsequence that converges to b.

## Problem 3

Let $(x_n)$ and $(y_n)$ be sequences in $\mathbb{R}^m$ such that both $\lim_n x_n$ and $\lim_n (x_n \cdot y_n)$ exist.

(1) When $m = 1$, prove that $\lim_n (y_n)$ need not exist.

**Ans:** Let $x_n = 1/n$ and $y_n = n$. Clearly $\lim_n (x_n) = 0$. Moreover, $x_n \cdot y_n = 1$, for all $n$, so that $\lim_n (x_n \cdot y_n) = 1$. However, $y_n$ increases without bound.

(2) When $m = 1$, prove that if $\lim_n x_n \neq 0$ then $\lim_n (y_n)$ exists.

**Ans:** In this answer, I'm going to assume that $x_n$ and $y_n$ live in $\mathbb{R}_+$. (The general proof is messier, with no additional insight.) Suppose that $\lim_n (x_n) = \alpha > 0$ while $\lim_n (x_n \cdot y_n) = \beta \geq 0$. We'll prove that $\lim_n (y_n) = \beta/\alpha$. Pick $\epsilon > 0$. We need to find $N \in \mathbb{N}$ such that $\forall n > N$, $y_n \in (\beta/\alpha - \epsilon, \beta/\alpha + \epsilon)$. Choose $N$ sufficiently large that for all $n > N$, $x_n \cdot y_n \in B(\beta, \delta_1)$, where $\delta_1 = \min\left(\frac{(\alpha\epsilon)^2}{\beta}, \frac{\beta\alpha\epsilon}{\beta+\alpha\epsilon}\right)$ while $x_n \in B(\alpha, \delta_2)$. where $\delta_2 = \min\left(\frac{\alpha^2\epsilon}{\beta}, \alpha\left[\frac{\alpha\epsilon}{\beta+\alpha\epsilon}\right]^2\right)$ Observe that for all $n > N$,

$$y_n > \frac{\beta - \frac{\alpha^2\epsilon^2}{\beta}}{x_n} > \frac{\beta - \frac{\alpha^2\epsilon^2}{\beta}}{\alpha + \frac{\alpha^2\epsilon}{\beta}} = \frac{\beta\left(1 - \left[\frac{\alpha\epsilon}{\beta}\right]^2\right)}{\alpha\left(1 + \frac{\alpha\epsilon}{\beta}\right)} = \frac{\beta\left(1 - \frac{\alpha\epsilon}{\beta}\right)}{\alpha} = \frac{\beta}{\alpha} - \epsilon$$

Also

$$y_n < \frac{\beta + \frac{\beta\alpha\epsilon}{\beta+\alpha\epsilon}}{x_n} < \frac{\beta + \frac{\beta\alpha\epsilon}{\beta+\alpha\epsilon}}{\alpha - \alpha\left[\frac{\alpha\epsilon}{\beta+\alpha\epsilon}\right]^2}$$

$$= \frac{\beta\left(1 + \frac{\alpha\epsilon}{\beta+\alpha\epsilon}\right)}{\alpha\left(1 - \left[\frac{\alpha\epsilon}{\beta+\alpha\epsilon}\right]^2\right)} = \frac{\beta}{\alpha\left(1 - \frac{\alpha\epsilon}{\beta+\alpha\epsilon}\right)} = \frac{\beta(\beta+\alpha\epsilon)}{\alpha\beta} = \frac{\beta}{\alpha} + \epsilon$$

(3) When $m = 2$, does $\lim_n x_n \neq 0$ imply that $\lim_n (y_n)$ necessarily exists. If so, prove it. If not, provide a counter-example.

**Ans:** When $m = 2$, $\lim_n x_n \neq 0$ does not imply that $\lim_n(y_n)$ necessarily exists. Let $x_n = (1/n, 1)$ and $y_n = (n, 1)$. We have $\lim_n x_n = (0, 1) \neq (0, 0)$ and $\lim_n(x_n \cdot y_n) = (1, 1)$. But $y_n$ does not have a limit.

## Problem 4

Consider the following function $\rho : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, defined by

$$\rho(x,y) \quad = \quad \begin{cases} \left|\frac{1}{x} - \frac{1}{y}\right| & \text{if } x,y > 0 \\ 0 & \text{if } x = y = 0 \\ 1/x & \text{if } y = 0 \\ 1/y & \text{if } x = 0 \end{cases}$$

(1) Verify that $\rho$ is a metric. (Checking the triangle inequality is a bit fiddly. You should first consider the case of $x, y > 0$. In my answer, I had to consider three separate cases; hopefully some of you will figure out something more elegant. Then suppose that either $x$ or $y$ is zero. I had to consider two cases here.)

**Ans:** Trivially, $\rho(x,y) \geq 0$, with $\rho(x,y) = 0$ iff $x = y$. Moreover, equally trivially $\rho(x,y) = \rho(y,x)$. So the only thing left to prove is that $\rho$ satisfies the triangle inequality, i.e., that $\forall x, y, z \in I$, $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$. First assume that $x, y > 0$ and, without loss of generality (w.l.o.g.) that $x < y$. I'll consider 3 cases:

(a) if $x \leq z \leq y$, then
$\rho(x,z) + \rho(z,y) = (1/x - 1/z) + (1/z - 1/y) = (1/x - 1/y) = \rho(x,y)$.

(b) if $z < x$, then
$\rho(x,z) + \rho(z,y) = (1/z - 1/x) + (1/z - 1/y) < (1/z - 1/x) + (1/x - 1/y) < (1/x - 1/y) = \rho(x,y)$.

(c) if $z > y$, then
$\rho(x,z) + \rho(z,y) = (1/x - 1/z) + (1/y - 1/z) < (1/x - 1/y) + (1/y - 1/z) < (1/x - 1/y) = \rho(x,y)$.

Now check that $\forall x, z > 0$, $\rho(x,0) \leq \rho(x,z) + \rho(z,0)$. There are two cases to consider

(a) if $z < x$, then
$\rho(x,z) + \rho(z,0) = (1/z - 1/x) + 1/z = 2/z - 1/x > 1/z > 1/x = \rho(x,0)$.

(b) if $z > x$, then
$\rho(x,z) + \rho(z,0) = (1/x - 1/z) + 1/z = 1/x = \rho(x,0)$.

(2) Identify (and verify) a necessary and sufficient condition for the set $I^\alpha = [\alpha, \infty) \subset \mathbb{R}$, $\alpha \geq 0$, to be closed in $\mathbb{R}_+$ under the metric $\rho$. If $I^\alpha$ is not closed for some $\alpha \geq 0$, write down the closure of $I^\alpha$ in $\mathbb{R}_+$ under the metric $\rho$.

**Ans:** $I^\alpha$ is closed iff $\alpha = 0$. If $\alpha = 0$, the complement of $I^\alpha$ in $\mathbb{R}_+$ is empty, so that the set has no boundary points. If $\alpha > 0$, the element $0 \in \mathbb{R}_+$ is a boundary point of $I^\alpha$ but does not belong to the set. To verify that 0 is indeed a boundary point, fix $\epsilon > 0$ and consider the point $y = \max(\alpha, 2/\epsilon)$. Clearly, $y \in I^\alpha$ and $y \in B(0, \epsilon)$, Also $0 \in B(0, \epsilon)$ and $0 \notin I^\alpha$. Therefore 0 satisfies the definition of a boundary point for $I^\alpha$. For $\alpha > 0$, the closure of $I^\alpha$ in $\mathbb{R}_+$ under the metric $\rho$ is $I^\alpha \cup \{0\}$.

(3) Identify and verify a necessary and sufficient condition for the set $I^\alpha = [\alpha, \infty) \subset \mathbb{R}$, $\alpha \geq 0$, to be bounded under the metric $\rho$ and the corresponding norm.

**Ans:** $I^\alpha$ is bounded iff $\alpha > 0$. If $\alpha > 0$, then for all $x \in I^\alpha$, $||x|| \leq ||\alpha|| = 1/\alpha$, so that $1/\alpha$ is a bound for $I^\alpha$. If $\alpha = 0$, then the set is unbounded, since for all $n \in \mathbb{N}$, $1/(n+1) \in I^0$ but $||1/(n+1)|| = n + 1 > n$.

(4) Consider the following two functions. (In each case, the metric on the range is Pythagorian.)

(a) $f : \mathbb{R}_+ \to \mathbb{R}_+$, defined by $f(x) = \begin{cases} 1/x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$. Prove that this function is continuous at zero when $\rho$ is the metric on the domain. (The metric on the range is Pythagorian.)

**Ans:** For any sequence $(x_n)$, a necessary condition for $x_n \to 0$ is that $(x_n)$ increases without bound. But in this case, the sequence $(f(x_n)) = (1/n)$ converges to zero. Hence we have proved: $x_n \to 0$ implies $(f(x_n)) \to 0 = f(0)$, establishing continuity of $f$ at zero.

(b) $f : Z_+ \to \mathbb{R}_+$, defined by $f(z) = 1/(z+1)$, where $Z_+$ denotes the nonnegative integers, i.e., $0, 1, 2, ....$

(i) Is this function continuous when $\rho$ is the metric on the domain?

**Ans:** No. In this case $f(0) = 1$. But as we have seen, the sequence $(z_n)$ defined by $z_n = n$ converges to zero. Moreover, $(f(z_n)) = (1/(n+1))$ converges to zero. Hence we have constructed a sequence $z_n \to 0$ s.t. $(f(z_n)) \to 0 \neq f(0)$.

(ii) What if the Pythagorian metric is the metric on the domain?

**Ans:** Yes. In this case, for arbitrary $z \in Z_+$, $(z_n)$ converges to $z$ only if $\exists N \in \mathbb{N}$ such that $\forall n > N$, $z_n = z$. But in this case, $\forall \epsilon > 0$, $\forall n > N$, $f(z_n) = f(z) \in B(f(z), \epsilon)$.

(5) When $\rho$ is the metric on $\mathbb{R}_+$, zero is not the most natural word to apply to the symbol "0". A more natural word would be ________________. Fill in the blank with one word: there is exactly one correct word. In *one sentence or less*, explain why this word is more natural.

**Ans:** The blank is "infinity." In this metric, the right hand side of the real line is where points bunch up; at the left hand side they spread out; that is, under this metric, the "direction" of the real line has been reversed, so that infinity plays the role that zero plays in the normal metric. (Note that that answer was one one sentence or less, since semicolons don't count.)