Fall 2008 **ARE211**

Problem Set #02

Second Analysis Problem Set

Due date: Sep 18

Problem 1

For the following problem, consider an arbitrary universe X, together with an arbitrary metric d defined on $X \times X$.

a) Prove that the union of arbitrarily many open sets is an open set.
b) Prove that the intersection of finitely many open sets is an open set.
c) Identify what part of your proof in part (b) would no longer hold if you would consider infinitely many sets.
d) Prove that the intersection of arbitrarily many closed sets is closed. (This is another very short proof, you just have to come up with a little trick. Hint: use DeMorgans formula $\cup_{i \in I} (X \setminus A_i) = X \setminus (\cap_{i \in I} A_i)$)

(Note: The union of two sets A, B is: $A \cup B = \{x \in X | x \in A \;\vee\; x \in B\}$). The intersection of two sets A, B is: $A \cap B = \{x \in X | x \in A \;\wedge\; x \in B\}$).

Problem 2

For the following problem, consider an arbitrary universe X, together with an arbitrary metric d defined on $X \times X$. Say whether each of the following statements is true or not. If the statements are true, prove them. If the statements are wrong, give a counter-example.

a) Every accumulation point of a set $S \subset X$ is a boundary point of S.
b) Every boundary point of a set $S \subset X$ is an accumulation point of S.
c) Every accumulation point s of a set $S \subset X$ that is not an element of S is a boundary point of S.
d) Every boundary point s of a set $S \subset X$ that is not an element of S is an accumulation point of S.

Problem 3

Prove that for any universe X, together with an arbitrary metric d defined on $X \times X$, the following is true for a set $S \subset X$.

$$\text{A set S is closed} \Leftrightarrow \text{cl(S)} = \text{S}.$$

Problem 4

Please use the Pythagorean metric. For each of the following subsets of $\mathbb{R}^2$: (i) sketch the set (ii) is it open? is it closed? is it compact? (iii) Give reasons for your negative answers to part (ii).

a) $\{(x_1, x_2) : x_1 = 0, x_2 \geq 0\}$
b) $\{(x_1, x_2) : 1 \leq x_1^2 + x_2^2 \leq 2\}$
c) $\{(x_1, x_2) : 1 \leq x_1 \leq 2\}$
d) $\{(x_1, x_2) : x_1 = 0 \vee x_2 = 0$, but not both $\}$

Problem 5

Which of the following statements are true *in general* for all sets $S \subset X$. (I.e. regardless of the chosen universe X and metric d). If they are true, prove them. If they are false, give a counter-example.

a) $int(cl(S)) = int(S)$
b) $cl(S) \cap S = S$
c) $cl(int(S)) = S$
d) $bd(cl(S)) = cl(bd(S))$

Problem 6

You will encounter proofs by induction all through the semester and therefore it's a good idea to practice the technique. Please prove by induction that $\forall n \in \mathbb{N}$

$$\sum_{k=1}^{n}(2k-1) = n^2$$