Fall 2008 **ARE211**

PROBLEM SET #01

FIRST ANALYSIS PROBLEM SET

DUE DATE: SEP 11

Problem 1

Please use the Pythagorean metric in the following problem. (The Pythagorian metric is another name for the d^2 metric, defined in class as $d^2(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$.)

a) Consider the sequence $x_n = 2 + \frac{(-1)^n}{n}$ defined on $\mathbb{R}$. Prove (i) that the sequence is a convergent sequence *using the definition of a convergent sequence* and show (ii) that the sequence is a Cauchy sequence *using the definition of a Cauchy sequence*.

b) Now consider the sequence $x_n = 2 + \frac{(-1)^n}{n}$ defined on $S = \mathbb{R} \setminus \{2\}$. Using your proof from part a) argue that it is still a Cauchy sequence in S. Prove that it is not a convergent sequence in S.

(Note: The set $A \setminus B$ is defined as: $A \setminus B = \{x | x \in A \ , \ x \notin B\}$).

Problem 2

a) Prove that a sequence x_n in X converges in the discrete metric if and only if there exists $\bar{x} \in X$ and a $N \in \mathbb{N}$ such that for all $n > N$, $x_n = \bar{x}$.

b) In class we showed that every Cauchy sequence in $\mathbb{R}$ with respect to the *Pythagorean metric* is also a convergent sequence in $\mathbb{R}$ with respect to the *Pythagorean metric*. Show that every Cauchy sequence in $\mathbb{R}$ with respect to the *discrete metric* is also a convergent sequence in $\mathbb{R}$ with respect to the *discrete metric*.

c) Problem 1 showed you that a Cauchy sequence that in defined on a strict subset of $\mathbb{R}$ does not have to converge in that subset. Again only considering the discrete metric, can we say that every Cauchy sequence defined on a subset $S \subset \mathbb{R}$ is also a convergent sequence in that subset. If yes, show why. If not, give a counter-example.

Problem 3

Show that every convergent sequence (in an arbitray universe X with respect to *any* metric defined on $X \times X$) is a Cauchy sequence under the same metric. (Hint: This proof is very short. Use the general definition of a metric).

Problem 4

For each of the following, draw and describe the ϵ-ball $B_d(\mathbf{x}, \epsilon|X)$ for some $\epsilon > 0$ around the point $\mathbf{x}$ in the specified metric d($\mathbf{x}$,$\mathbf{y}$) and universe X. (In part (d) you might not be able to draw it, so just sketch parts of it). Give a brief explanation of why the ϵ-ball looks the way it does. For (d) and (e), consider two cases, one where $\epsilon < 1$ and another where $\epsilon > 1$.

Part	**x**	d(**x**,**y**)	X
(a)	3	$\lvert x - y\rvert$	$(-\infty, 3]$
(b)	(2,1)	$max\{\lvert x_1 - y_1\rvert, \lvert x_2 - y_2\rvert\}$	$\mathbb{R}^2$
(c)	(1,2)	$\sum_{i=1}^{2} \lvert x_i - y_i\rvert$	$\mathbb{R}^2$
(d)	2	discrete metric	Rationals $\mathbb{Q}$
(e)	(2,2)	Pythegorean metric	$\mathbb{Z} \times \mathbb{Z}$ where $\mathbb{Z}$ are the integers

Problem 5

Show that each subset S of an arbitrary universe X is an open set in X under the discrete metric.

Problem 6

Fix $a, b, c \in \mathbb{R}$ with $a < b < c$. Consider the following two subsets of $\mathbb{R}^2$:

$$S_h = \{\mathbf{x} \in \mathbb{R}^2 | a < x_1 < b \ ; \ x_2 = c\}$$

$$S_v = \{\mathbf{x} \in \mathbb{R}^2 | x_1 = a \ ; \ b < x_2 < c\}$$

Loosely speaking, S_h is a line segment parallel to the horizontal axis and S_v is a line segement parallel to the vertical axis.

- You know from Problem 5 that all S_h and S_v are open sets under the discrete metric.

a) Show that neither S_h nor S_v are open sets under the Pythagorean metric.

b) Is it possible that all S_h are open sets and all S_v are *not* open sets under the same metric. If yes, then look far and wide and give an example of such a metric. If not, prove why it isn't possible.

Since we want to keep problem sets shorter, there are two more optional problems below. We do *strongly* recommend that you do them. I will grade them and record that you did them, but the points will not be a part of your grade unless you are a "border-line" case when it comes time to calculate the final grades.

Optional Problem 1

Prove the following: Given $S \subset \mathbb{R}$, $b \in \mathbb{R}$ is a greatest lower bound (infimum) of S iff b is a lower bound for S and $\forall\ \epsilon > 0, \exists s \in S$, such that $s - b < \epsilon$. A very similar proof is in the notes; please try this first on your own without referring to that proof.

Optional Problem 2

Recall the definition of point-wise convergence we gave in class. A sequence of functions f_n converges point-wise to a function f on a set X in the metric d if $\forall x \in X$, given $\epsilon > 0\ \exists N(x, \epsilon) \in \mathbb{N}$ such that $n > N$ implies $d(f_n(x), f(x)) < \epsilon$.

This definition implies that N depends on the epsilon *and* on the x. As we discussed in section, it may not be possible to find an N that works for *every* x simultaneously. If you succeed in finding such an N, you have uniform convergence.

We define uniform convergence as: A sequence of functions f_n converges uniformly to a function f on the set X in the metric d if $\forall\ \epsilon > 0\ \exists\ N(\epsilon) \in \mathbb{N}$ s.t. $\forall\ x \in X$, $n > N$ implies $d(f_n(x), f(x)) < \epsilon$.

For each of the following state whether they are true or not. If they are correct, prove them. If they are false, give a counter-example.

a) For a given universe X and a metric d, every sequence of functions that converges uniformly also converges point-wise.
b) For a given universe X and the Pythagorian metric, every point-wise convergent sequence of functions also converges uniformly.
c) For a given universe X and the discrete metric, every point-wise convergent sequence of functions also converges uniformly.