Problem Set #01

First Analysis Problem Set

Due date: Sep 12

Problem 1

Please use the Pythagorean metric in the following problem. (The Pythagorian metric is another name for the d^2 metric, defined in class as $d^2(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$.)

a) Consider the sequence $x_n = 2 + \frac{(-1)^n}{n}$ defined on $\mathbb{R}$. Prove (i) that the sequence is a convergent sequence *using the definition of a convergent sequence* and show (ii) that the sequence is a Cauchy sequence *using the definition of a Cauchy sequence.*

b) Now consider the sequence $x_n = 2 + \frac{(-1)^n}{n}$ defined on $S = \mathbb{R} \setminus \{2\}$. Using your proof from part a) argue that it is still a Cauchy sequence in S. Prove that it is not a convergent sequence in S.

(Note: The set $A \setminus B$ is defined as: $A \setminus B = \{x | x \in A \ , \ x \notin B\}$).

Ans: Recall the following definitions for convergence:

(C) *Convergent Sequence:* A sequence x_n in X converges to an element $x \in X$ in the metric d if $\forall \ \epsilon > 0 \ \exists N \in \mathbb{N}$ such that $\forall n > N$ we have $d(x_n, x) < \epsilon$.

(CA) *Cauchy Sequence:* A sequence x_n in X is a Cauchy sequence under the metric d if $\forall \ \epsilon > 0 \ \exists N \in \mathbb{N}$ such that $\forall n, m > N$ we have $d(x_n, x_m) < \epsilon$.

Now let's apply those definitions to the sequence $x_n = 2 + \frac{(-1)^n}{n}$ using the Pythagorian metric.

(a) (i) The sequence converges to $2 \in X$ as $\forall \ \epsilon > 0$ pick $N = \lceil \frac{1}{\epsilon} \rceil \in \mathbb{N}$ and $\forall n > N$

$d(x_n, x) = |2 + \frac{(-1)^n}{n} - 2| = |\frac{(-1)^n}{n}|$

$= \frac{1}{n}$ (The absolute value of $(-1)^n$ is 1)

$< \frac{1}{N}$ (Given $n > N$)

$\leq \frac{1}{\frac{1}{\epsilon}} = \epsilon$ (Given $N = \lceil \frac{1}{\epsilon} \rceil$)

(ii) The sequence is a Cauchy sequence as $\forall \ \epsilon > 0$ pick $N = \lceil \frac{2}{\epsilon} \rceil \in \mathbb{N}$ and $\forall n, m > N$

$d(x_n, x_m) = |2 + \frac{(-1)^n}{n} - 2 - \frac{(-1)^m}{m}| = |\frac{(-1)^n}{n} + \frac{-(-1)^m}{m}|$

$\leq |\frac{(-1)^n}{n}| + |\frac{-(-1)^m}{m}|$

(The triangle inequelity of the absolute value)

$= \frac{1}{n} + \frac{1}{m}$

(The absolute value of $(-1)^n$ and $-(-1)^m$ is 1)

$< \frac{1}{N} + \frac{1}{N}$

(Given $n, m > N$)

$= \frac{2}{N}$

$\leq \frac{2}{\frac{2}{\epsilon}} = \epsilon$

(Given $N = \lceil \frac{2}{\epsilon} \rceil$)

(b) (i) *Convergent sequence:* One easy way out might be to assume it converges to a point other than 2 and then argue that this is a violation of problem 6 on problem set 2 (i.e. that a sequence converges to at most one point). However, be careful as you are dealing with two different universes here.

It might be a good exercise to show this part using the definition (C). A sequence does *not* converge to a point $x \in X$ in the metric d if $\exists\ \epsilon > 0$ s.t. $\forall N \in \mathbb{N}\ \exists n > N$ with $d(x_n, x) \geq \epsilon$.

Every point x in the universe $X = \mathbb{R} \setminus \{2\}$ can be written as $x = 2 + \delta,\ \delta \neq 0$ We know have to consider two cases:

case 1) Let $\delta > 0$. We now show that this can't be a limit: Let $\epsilon = \delta$ and $\forall N \in \mathbb{N}\ \exists n = 2N + 1 > N$ with

$d(x_n, x) = |2 + \frac{(-1)^n}{n} - 2 - \delta| = |\frac{(-1)^{2N+1}}{2N+1} - \delta|$

$= |\frac{-1}{2N+1} - \delta|$

(As $2N + 1$ is an odd number)

$= \frac{1}{2N+1} + \delta$

(As both $\frac{-1}{2N+1}$ and $-\delta$ are negative)

$> \delta$

(As $\frac{1}{2N+1}$ is positive)

$= \epsilon$

case 2) Let $\delta < 0$. We now show that this can't be a limit: Let $\epsilon = -\delta$ and $\forall N \in \mathbb{N}\ \exists n = 2N > N$ with

$d(x_n, x) = |2 + \frac{(-1)^n}{n} - 2 - \delta| = |\frac{(-1)^{2N}}{2N} - \delta|$

$= |\frac{1}{2N} - \delta|$

(As $2N$ is an even number)

$= \frac{1}{2N} - \delta$

(As both $\frac{1}{2N}$ and $-\delta$ are positive)

$> -\delta$

(As $\frac{1}{2N}$ is positive)

$= \epsilon$

(ii) *Cauchy sequence:* The proof that x_n is a Cauchy sequence in part (a) does not depend on whether the limit x is in the set or not. Clearly all points of the sequence $\{x_n\}$ lie in $S = \mathbb{R} \setminus \{2\}$ and the same proof thus holds.

Problem 2

a) Prove that a sequence x_n in X converges in the discrete metric if and only if there exists $\bar{x} \in X$ and a $N \in \mathbb{N}$ such that for all $n > N$, $x_n = \bar{x}$.
b) In class we showed that every Cauchy sequence in $\mathbb{R}$ with respect to the *Pythagorean metric* is also a convergent sequence in $\mathbb{R}$ with respect to the *Pythagorean metric*. Show that every Cauchy sequence in $\mathbb{R}$ with respect to the *discrete metric* is also a convergent sequence in $\mathbb{R}$ with respect to the *discrete metric*.
c) Problem 1 showed you that a Cauchy sequence that in defined on a strict subset of $\mathbb{R}$ does not have to converge in that subset. Again only considering the discrete metric, can we say that every Cauchy sequence defined on a subset $S \subset \mathbb{R}$ is also a convergent sequence in that subset. If yes, show why. If not, give a counter-example.

Ans:

(A) A sequence converges in the discrete metric, i.e. there $\exists \bar{x} \in X$ and $\forall\ \epsilon > 0\ \exists N_1 \in \mathbb{N}$ such that $\forall n > N_1$ we have $d(x_n, \bar{x}) < \epsilon$
(B) $\exists N_2 \in \mathbb{N}$ such that $\forall n > N_2$ we have $x_n = \bar{x}$.
(C) A sequence x_n in X is a Cauchy sequence under the discrete metric d if $\forall\ \epsilon > 0\ \exists N_3 \in \mathbb{N}$ such that $\forall n, m > N_3$ we have $d(x_n, x_m) < \epsilon$.

In the following, d always stands for the *discrete metric*.

part a) We have to show that $A \Leftrightarrow B$.

"$\Leftarrow$"
(1) We are given (B), i.e., $\exists N_2 \in \mathbb{N}$ s.t. $\forall n > N_2$ we have $x_n = \bar{x}$.
(2) A sequence lives in the universe and thus $\bar{x} \in X$
(3) (1) and (2) imply that $\exists \bar{x} \in X$ and $\forall\ \epsilon > 0\ \exists N_1 = N_2$ s.t. $d(x_n, \bar{x}) = d(\bar{x}, \bar{x}) = 0 < \epsilon$

"$\Rightarrow$"
(1) We are given (A), i.e., $\exists \bar{x} \in X$ and $\forall\ \epsilon > 0\ \exists N_1 \in \mathbb{N}$ such that $\forall n > N_1$ we have $d(x_n, \bar{x}) < \epsilon$.
(2) Now choose $\epsilon = \frac{1}{2}$ and we know from (1) that $\forall n > N_2 = N_1 : d(x_n, \bar{x}) < \frac{1}{2}$
(3) Given the discrete metric two points can only have a distance of less than $\frac{1}{2}$ if they are equal, i.e. $x_n = \bar{x}\ \forall n > N_2$.

part b) We have to show that $C \Rightarrow A$.

(1) We are given (C), i.e., $\forall\ \epsilon > 0\ \exists N_3 \in \mathbb{N}$ such that $\forall n, m > N_3$ we have $d(x_n, x_m) < \epsilon$
(2) Define $\bar{x} = x_{N_3+1}$.
(3) A sequence lives in the universe and thus $\bar{x} \in X$
(4) Now choose $\epsilon = \frac{1}{2}$ and we know from (1) and (2) that $\forall n > N_2 = N_3 : d(x_n, \bar{x}) < \frac{1}{2}$
(5) Given the discrete metric two points can only have a distance of less than $\frac{1}{2}$ if they are equal, i.e. $\forall n > N_2\ x_n = \bar{x}$.
(6) In (5) we have shown that (B) is true. Given part (a) we now that if (B) is true than (A) must be true too.

part c) True: The proof in part (b) did not utilize the fact that the universe was $\mathbb{R}$. The crucial difference to the Pythagorean metric is in steps (2) and (3) in part (b), i.e., under the discrete metric a sequence can only converge to an *element* of the sequence which automatically lies in the universe.

Problem 3

Show that every convergent sequence (in an arbitray universe X with respect to *any* metric defined on $X \times X$) is a Cauchy sequence under the same metric. (Hint: This proof is very short. Use the general definition of a metric).

Ans: Recall the following definitions for convergence:

(A) *Convergent Sequence:* A sequence x_n in X converges to an element $x \in X$ in the metric d if $\forall\ \delta > 0\ \exists N_1 \in \mathbb{N}$ such that $\forall n > N_1$ we have $d(x_n, x) < \delta$.

(B) *Cauchy Sequence:* A sequence x_n in X is a Cauchy sequence under the metric d if $\forall\ \epsilon > 0\ \exists N_2 \in \mathbb{N}$ such that $\forall n, m > N_2$ we have $d(x_n, x_m) < \epsilon$.

We have to show that $A \Rightarrow B$.

(1) $\forall\ \epsilon > 0$ define $\delta = \frac{\epsilon}{2} > 0$

(2) Using (A) we know that $\exists N_1 \in \mathbb{N}$ s.t. $\forall n > N_1$ we have $d(x_n, x) < \delta$

(3) We therefore know that $\forall\ \epsilon > 0\ \exists N_2 = N_1$ s.t. $\forall n, m > N_2$

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x) + d(x, x_m) && \text{(Triangle inequality for d)} \\ &= d(x_n, x) + d(x_m, x) && \text{(Symmetry of d)} \\ &< \delta + \delta && \text{(By (2))} \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} && \text{(By (1))} \\ &= \epsilon \end{aligned}$$

Problem 4

For each of the following, draw and describe the ϵ-ball $B_d(\mathbf{x}, \epsilon|X)$ for some $\epsilon > 0$ around the point $\mathbf{x}$ in the specified metric d($\mathbf{x}$,$\mathbf{y}$) and universe X. (In part (d) you might not be able to draw it, so just sketch parts of it). Give a brief explanation of why the ϵ-ball looks the way it does. For (d) and (e), consider two cases, one where $\epsilon < 1$ and another where $\epsilon > 1$.

Part	**x**	d(**x**,**y**)	X
(a)	3	$\|x - y\|$	$(-\infty, 3]$
(b)	(2,1)	$max\{\|x_1 - y_1\|, \|x_2 - y_2\|\}$	$\mathbb{R}^2$
(c)	(1,2)	$\sum_{i=1}^{2} \|x_i - y_i\|$	$\mathbb{R}^2$
(d)	2	discrete metric	Rationals $\mathbb{Q}$
(e)	(2,2)	Pythegorean metric	$\mathbb{Z} \times \mathbb{Z}$ where $\mathbb{Z}$ are the integers

Ans: The ϵ-ball for part (a) and $\epsilon = 1$ is the solid line in figure 1. Note that the ϵ-ball can not include points bigger than 3 as the universe is $(-\infty, 3]$. Part of the universe is displayed as the dotted line. The ϵ-ball for part (d) and $\epsilon \in \{\frac{1}{2}, 2\}$ are displayed in figure 2. When $\epsilon = \frac{1}{2}$, the ϵ-ball consists only of the point $\{2\}$. In case $\epsilon = 2$, the ϵ-ball contains all rationals. (The picture depicts 100 rationals that have randomly been drawn) The ϵ-ball for part (b) and (c) and $\epsilon = 1$ are displayed in figure 3. Note that the ϵ-ball does *not* include the dashed border.

FIGURE 1. Balls for part (a) where epsilon = 1

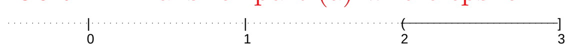


FIGURE 2. Balls for part (d) where epsilon = $\frac{1}{2}$ (left) and epsilon = 2 (right)

The ϵ-ball for part (e) and $\epsilon \in \{\frac{1}{2}, 2\}$ are displayed in figure 4. When $\epsilon = \frac{1}{2}$, the ϵ-ball consists only of the point $\{(2, 2)\}$. In case $\epsilon = 2$, the ϵ-ball contains the points $\{(1, 1); (1, 2); (1, 3); (2, 1); (2; 2); (2; 3); (3; 1); (3; 2); (3; 3)\}$. (The universe is restricted to the integers)

Figure 3. Balls for part b (left) and c (right) where epsilon = 1

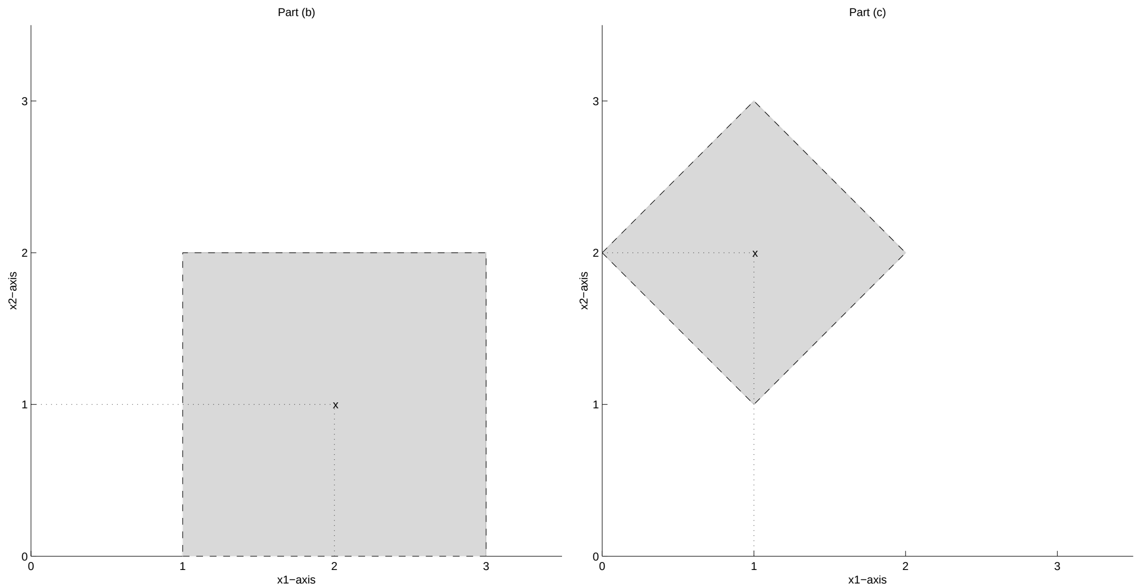
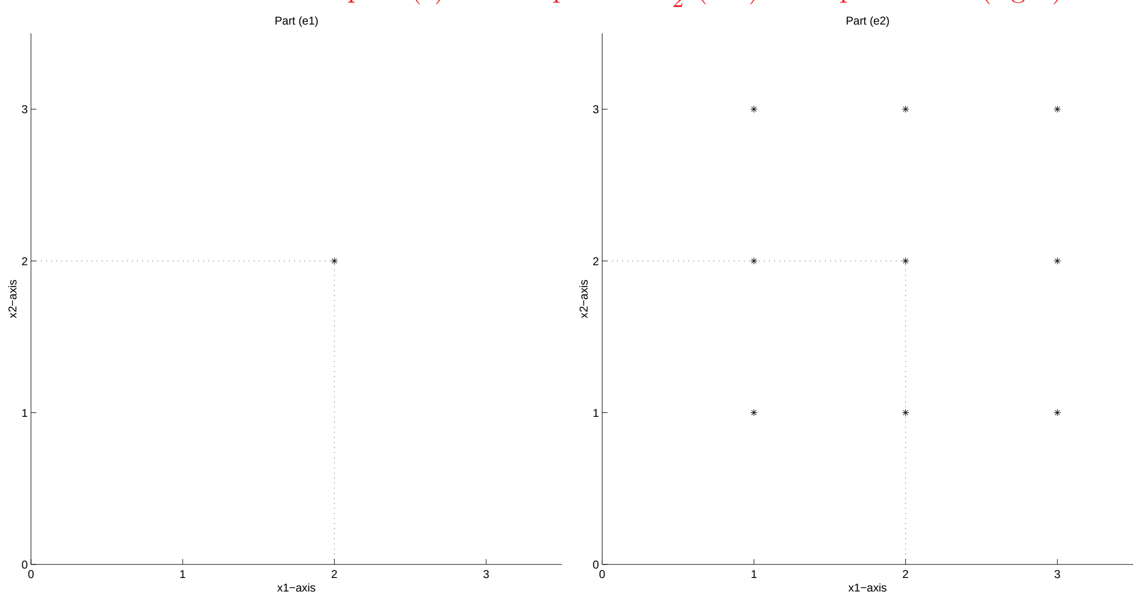


Figure 4. Balls for part (e) where epsilon = $\frac{1}{2}$ (left) and epsilon = 2 (right)



Problem 5

Show that each subset S of an arbitrary universe X is an open set in X under the discrete metric.

Ans: Recall the definition of an open set:

A set $S \subset X$ is said to be open in X w.r.t. a metric d if $\forall x \in S\ \exists \epsilon > 0$ such that $B_d(x, \epsilon) = \{y \in X | d(x, y) < \epsilon\} \subset S$.

We have to show: "d is the discrete metric" $\Rightarrow$ Every set $S \subset X$ is open.

(1) Consider an arbitrary set S: $\forall x \in S$ note that by construction $x \in S$

(2) Choose $\epsilon = \frac{1}{2}$. Then the ball $B_d(x, \frac{1}{2}) = \{y \in X | d(x, y) < \frac{1}{2}\} = \{x\}$ (I.e., the ball consists only of the point x which after (1) is an element of S)

(3) Hence by (1) and (2) $\forall x \in S\ \exists \epsilon = \frac{1}{2} > 0$ such that the ball $B_d(x, \epsilon = \frac{1}{2}) = \{y \in X | d(x, y) < \frac{1}{2}\} \subset S$.

Problem 6

Fix $a, b, c \in \mathbb{R}$ with $a < b < c$. Consider the following two subsets of $\mathbb{R}^2$:

$$S_h = \{\mathbf{x} \in \mathbb{R}^2 | a < x_1 < b \ ; \ x_2 = c\}$$

$$S_v = \{\mathbf{x} \in \mathbb{R}^2 | x_1 = a \ ; \ b < x_2 < c\}$$

Loosely speaking, S_h is a line segment parallel to the horizontal axis and S_v is a line segement parallel to the vertical axis.

- You know from Problem 5 that all S_h and S_v are open sets under the discrete metric.

a) Show that neither S_h nor S_v are open sets under the Pythagorean metric.

b) Is it possible that all S_h are open sets and all S_v are *not* open sets under the same metric. If yes, then look far and wide and give an example of such a metric. If not, prove why it isn't possible.

Ans: Recall the definition of an open set:

A set $S \subset X$ is said to be open in X w.r.t. a metric d if $\forall x \in S \ \exists \epsilon > 0$ such that $B_d(x, \epsilon) = \{y \in X | d(x, y) < \epsilon\} \subset S$.

Hence, a set is *not* open if

A set $S \subset X$ is *not* open in X w.r.t. a metric d if $\exists x \in S$ and $\forall \epsilon > 0$ the ball $B_d(x, \epsilon) = \{y \in X | d(x, y) < \epsilon\} \not\subset S$.

Now let's examine the S_h and the S_v

a) In this part we are using the Pythagorian metric

S_h:

(1) Consider the point $\bar{x} = (\frac{a+b}{2}, c)$ which lies in S_h

(2) $\forall \epsilon > 0$ the ball $B_d(\bar{x}, \epsilon)$ includes the point $y = (\frac{a+b}{2}, c + \frac{\epsilon}{2})$ as $d(\bar{x}, y) = \frac{\epsilon}{2} < \epsilon$.

(3) The point y is not an element of S_h as $y_2 \neq \bar{x}_2$ and hence the ball $B_d(\bar{x}, \epsilon)$ is *not* a subset of S_h

(4) (1) to (3) tells us that S_h is *not* open.

S_v: (1) Consider the point $\bar{x} = (a, \frac{b+c}{2})$ which lies in S_v

(2) $\forall \epsilon > 0$ the ball $B_d(\bar{x}, \epsilon)$ includes the point $y = (a + \frac{\epsilon}{2}, \frac{b+c}{2})$ as $d(\bar{x}, y) = \frac{\epsilon}{2} < \epsilon$.

(3) The point y is not an element of S_v as $y_1 \neq \bar{x}_1$ and hence the ball $B_d(\bar{x}, \epsilon)$ is *not* a subset of S_v

(4) (1) to (3) tells us that S_v is *not* open.

b) Let $\rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ be any metric on $\mathbb{R}$ that is equivalent to the Pythagorian metric. Now consider the metric $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_+$ defined by

$$d(x, y) = \begin{cases} \min[\rho(x_1, y_1), 1] & \text{if } x_2 = y_2 \\ 1 & \text{if } x_2 \neq y_2 \end{cases}$$

Essentially this metric is the discrete metric on the second dimension, combined with a regular metric on the first. Before showing that this metric does what we want it to do, we need to verify that it is indeed a metric. I'll just check the triangle inequality; the other properties are clearly satisfied. There are two cases we need to consider:

- $x_2 = y_2$, so that $d(x, y) = \min[\rho(x_1, y_1), 1] \leq 1$. Pick $z \in \mathbb{R}^2$. If $z_2 \neq x_2$, then $d(x, z) + d(z, y) = 2 > d(x, y)$. Assume therefore that $z_2 = x_2$. If $d(x, z) + d(z, y) \geq 1$, then $d(x, z) + d(z, y) \geq d(x, y)$. If $d(x, z) + d(z, y) < 1$, then $1 > d(x, z) + d(z, y) = \rho(x_1, z_1) + \rho(z_1, y_1) \geq \rho(x_1, y_1)$, using the fact that ρ is a metric on $\mathbb{R}$.
- $x_2 \neq y_2$, so that $d(x, y) = 1$. Pick $z \in \mathbb{R}_2$; if $z_2 = x_2$ then $x_2 \neq y_2$. Hence $d(x, z) + d(z, y) \geq 1$.

We know show that :nder the metric d, all S_h are open, and all S_v are *not* open in $\mathbb{R}^2$.

S_h: (1) $\forall \bar{x} \in S_h$ choose $\bar{\epsilon} = \frac{1}{2}$ min $\{\bar{x}_1 - a \ ; \ b - \bar{x}_1 \ ; \ 1\}$

(2) $B_d(\bar{x}, \bar{\epsilon}) = \{\mathbf{x} \in \mathbb{R}^2 | \bar{x}_1 - \bar{\epsilon} < x_1 < \bar{x}_1 + \bar{\epsilon}; x_2 = \bar{x}_2\} \subset S_h$

(3) (1) and (2) tells us that S_h is open.

S_v: (1) Consider the point $\bar{x} = (a, \frac{b+c}{2})$ which lies in S_v

(2) Fix $\epsilon > 0$ and pick $z \in B_\rho(a, \epsilon | \mathbb{R})$, $z \neq a$. The ball $B_d(\bar{x}, \epsilon)$ includes the point $y = (z, \frac{b+c}{2})$.

(3) The point y is not an element of S_v as $y_1 \neq \bar{x}_1$ and hence the ball $B_d(\bar{x}, \epsilon)$ is *not* a subset of S_v

(4) (1) to (3) tells us that S_v is *not* open.

Since we want to keep problem sets shorter, there are two more optional problems below. We do *strongly* recommend that you do them. I will grade them and record that you did them, but the points will not be a part of your grade unless you are a "border-line" case when it comes time to calculate the final grades.

Optional Problem 1

Prove the following: Given $S \subset \mathbb{R}$, $b \in \mathbb{R}$ is a greatest lower bound (infimum) of S iff b is a lower bound for S and $\forall\ \epsilon > 0, \exists s \in S$, such that $s - b < \epsilon$. A very similar proof is in the notes; please try this first on your own without referring to that proof.

Ans: See the proof in the lecture notes

Optional Problem 2

Recall the definition of point-wise convergence we gave in class. A sequence of functions f_n converges point-wise to a function f on a set X in the metric d if $\forall x \in X$, given $\epsilon > 0\ \exists N(x, \epsilon) \in \mathbb{N}$ such that $n > N$ implies $d(f_n(x), f(x)) < \epsilon$.

This definition implies that N depends on the epsilon *and* on the x. As we discussed in section, it may not be possible to find an N that works for *every* x simultaneously. If you succeed in finding such an N, you have uniform convergence.

We define uniform convergence as: A sequence of functions f_n converges uniformly to a function f on the set X in the metric d if $\forall\ \epsilon > 0\ \exists\ N(\epsilon) \in \mathbb{N}$ s.t. $\forall\ x \in X$, $n > N$ implies $d(f_n(x), f(x)) < \epsilon$.

For each of the following state whether they are true or not. If they are correct, prove them. If they are false, give a counter-example.

a) For a given universe X and a metric d, every sequence of functions that converges uniformly also converges point-wise.
b) For a given universe X and the Pythagorian metric, every point-wise convergent sequence of functions also converges uniformly.
c) For a given universe X and the discrete metric, every point-wise convergent sequence of functions also converges uniformly.

Ans:

a) **True**. Uniform convergence is a stronger requirement than pointwise convergence, so that if something satisfies the former it satisfies the latter. Specifically, assume that (f_n) converges uniformly. From the definition I gave in the problem set, we know $\forall\ \epsilon > 0\ \exists\ \bar{N}(\epsilon) \in \mathbb{N}$ s.t. $\forall\ x \in X$, $n > \bar{N}(\epsilon)$ implies $d(f_n(x), f(x)) < \epsilon$. Now fix $x \in X$ and $\epsilon > 0$. We need to find $N(x, \epsilon) \in \mathbb{N}$ such that $n > N(x, \epsilon)$ implies $d(f_n(x), f(x)) < \epsilon$. Let $N(x, \epsilon) = \bar{N}(\epsilon)$.
b) **False**. For an example of a sequence of functions that converges pointwise but not uniformly, consider the sequence I defined in the first lecture, i.e., $\{f_1, f_2, ...f_n...\}$, where $f_n =$

$\begin{cases} -1 & \text{if } x \leq -1/n \\ nx & \text{if } -1/n < x < 1/n \\ 1 & \text{if } x \geq 1/n \end{cases}$ It's easy to check that this sequence converges pointwise to the function $f_0 = \begin{cases} -1 & \text{if } x < 0 \\ nx & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$. (I won't write out the details.) But it doesn't converge uniformly. To see this let $\epsilon = 1/4$, pick $n \in \mathbb{N}$ and let $x = 1/2n$. We have $f_n(x) = n/2n = 0.5$, while $f_0(x) = 1$, so that $d(f_n(x), f(x)) > 0.25 = \epsilon$.

c) **False** (I'd always thought this was true, but now I come to write down the answer, it clearly isn't.) The following is a modified version of the sequence of functions in the answer to b). Let $f_n = \begin{cases} 0 & \text{if } x < 1/n \\ 1 & \text{if } x \geq 1/n \end{cases}$. It's easy to check that this sequence converges pointwise to the function $f_0 = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$, i.e., for $\epsilon > 0$ and $x > 0$, pick $N(x, \epsilon) > 1/x$ so that for $n > N(x, \epsilon)$, $f_n(x) = 1 = f_0(x)$, and $d(f_n(x), f(x)) = 0 < \epsilon$. On the the other hand, for $\epsilon = 0.5$ and $n \in \mathbb{N}$, pick (again) $x = 1/2N$ and note that $f_n(x) = 0 \neq f_0(x)$, so that $d(f_n(x), f(x)) = 1 > \epsilon$.