ARE211, Fall 2009

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Contents

6. Nonlinear Programming Problems and the Kuhn Tucker conditions (cont)

Key points:

(1) The relationship between the KKT conditions and the Lagrangian: they are exactly the same. The Lagrangian is a "long-hand" way to get to the same place.

(2) Interpretation of the Lagrangian multiplier: λ_j is the rate at which the maximized value of the objective increases as the j'th constraint is relaxed.

  (a) multipliers increases with the length of the gradient of the objective at the solution

  (b) the j'th multiplier *decreases* with the length of the gradient of the j'th constraint at the solution

(3) Lots of practice at computation

6.4. **KT conditions and the Lagrangian approach**

The version of the KT conditions that we've seen in the previous lectures is intuitive and lends itself to diagramatic explanations, etc. For some people, it's a bit abstract however. To actually solve lagrangian problems, most people typically proceed by setting up a *Lagrangian* objective, for which they compute first order conditions. What's important to understand is that the Lagrangian route is a long way around to end up at *exactly* the same place that we've gotton to in the preceeding lectures. the FOC for the Lagragian is nothing other than a line-by-line representation of the KKT necessary conditions we've expressed previously in vector (or matrix) form.

The Lagrangian function for the canonical NPP is as follows:

$$L(\mathbf{x}, \boldsymbol{\lambda}) \quad = \quad f(\mathbf{x}) + \boldsymbol{\lambda}(\mathbf{b} - \mathbf{g}(\mathbf{x})) \quad = \quad f(\mathbf{x}) + \sum_{j=1}^{m} \lambda_j (b_j - g^j(\mathbf{x}))$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{b} \in \mathbb{R}^m$. Note that $\boldsymbol{\lambda}$ is treated just like $\mathbf{x}$, i.e., as a variable to be optimized. The first order conditions for an extremum of L on the space $\mathbb{R}^n \times \mathbb{R}^m_+$ are:

$$\text{for } i = 1, ...n, \partial L(\bar{\mathbf{x}}, \bar{\boldsymbol{\lambda}})/\partial x_i = 0; \quad \text{for } j = 1, ...m, \partial L(\bar{\mathbf{x}}, \bar{\boldsymbol{\lambda}})/\partial \lambda_j \geq 0 \text{ and } \bar{\lambda}_j \partial L(\bar{\mathbf{x}}, \bar{\boldsymbol{\lambda}})/\partial \lambda_j = 0. \quad (1)$$

Note well: this specification of the Lagrangian first order conditions differs from many textbooks, eg., Chiang, which write the FOC w.r.t. $\mathbf{x}$ as $\partial L(\bar{\mathbf{x}}, \bar{\boldsymbol{\lambda}})/\partial x_i \leq 0$. The reason for the difference is that these textbooks are adding an additional restriction on their specification of the NPP, i.e., that $\mathbf{x} \geq 0$. As you recall, my specification of the canonical problem treats the requirement that $\mathbf{x} \geq 0$ as just n more inequality constraints, i.e., $\mathbf{x} \geq 0$.

You'll observe that I didn't say FOC for a maximum of L; technically, what we're looking for is a maximum of L w.r.t. $\mathbf{x}$ and a *minimum* of L w.r.t. λ, subject to the additional constraint that the λ vector is nonnegative. It would be highly appropriate to ask, at this point, *why on earth would anybody ever consider looking for such a weird object* as a max w.r.t. $\mathbf{x}$ and a min w.r.t. $\boldsymbol{\lambda}$ s.t. $\boldsymbol{\lambda} \geq 0$? It's not highly intuitive. In fact, everybody always just talks about "the first order conditions for the Lagrangian" and nobody bothers to wonder what it means, except in classes like this. I strongly recommend not bothering to ask this highly appropriate question; instead, think of the Lagrangian is nothing but a construction that delivers us the KTT conditions, i..e,

Theorem: A vector $(\bar{\mathbf{x}}, \bar{\boldsymbol{\lambda}})$ satisfies the KTT conditions iff it satisfies (1)

To prove the theorem, recall the part of the KKT condition which states

$$\bigtriangledown f(\bar{\mathbf{x}})^T = \bar{\boldsymbol{\lambda}}^T J\mathbf{g}(\bar{\mathbf{x}}) \tag{2}$$

On the other hand, writing out $\partial L(\bar{\mathbf{x}}, \bar{\boldsymbol{\lambda}})/\partial x_i = 0$, we have

$$\partial f(\bar{\mathbf{x}})/\partial x_i = \sum_{j=1}^{m} \bar{\lambda}_j \cdot \partial g^j(\bar{\mathbf{x}})/\partial x_i \tag{3}$$

The left hand side of (3) is the i'th element of the vector $\bigtriangledown f(\bar{\mathbf{x}})$. The right hand side is the inner product of $\bar{\boldsymbol{\lambda}}$ with the i'th row of the matrix $Jg(\mathbf{x})$. Thus, the first order conditions for the x_i's are simply a long-hand way of writing out what (2) states concisely in vector notation. Now consider the m conditions on the λ_j's, i.e., $\partial L(\bar{\mathbf{x}}, \bar{\boldsymbol{\lambda}})/\partial \lambda_j \geq 0$, or $b_j \geq g^j(\mathbf{x})$. Taken together these m conditions state that $\mathbf{x}$ belongs to the intersection of the lower contour sets of the g^j's corresponding to the b_j's. Finally, consider the condition $\bar{\lambda}_j \partial L(\bar{\mathbf{x}}, \bar{\boldsymbol{\lambda}})/\partial \lambda_j = 0$, which is known as a *complementary slackness* condition. This condition will be satisfied iff *either* $\bar{\lambda}_j = 0$. *or* $b_j = g^j(\mathbf{x})$ *or* both equalities are satisfied simultaneously.

An important property of the Lagrangian specification is that the value of the Lagrangian function at the solution to the NPP is equal to the maximized value of the objective function on the constraint set. In symbols:

Theorem: At a solution $(\bar{\mathbf{x}}, \bar{\boldsymbol{\lambda}})$ to the NPP, $L(\bar{\mathbf{x}}, \bar{\boldsymbol{\lambda}}) = f(\bar{\mathbf{x}})$.

To see that this is correct note that $L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \boldsymbol{\lambda}(\mathbf{b} - \mathbf{g}(\mathbf{x}))$. Each component of the inner product $\boldsymbol{\lambda}(\mathbf{b} - \mathbf{g}(\mathbf{x}))$ is zero, since *either* $(b_j - g^j(\mathbf{x}))$ is zero *or* λ_j is zero. Hence the entire second term on the right hand side of the Lagrangian is zero.

6.5. **Interpretation of the Lagrange Multiplier**

The Lagrangian λ_j has an interpretation that proves to be important in a wide variety of economic applications.

- it is a measure of how much bang for the buck you get when you relax the j'th constraint. That is, if you *increase* b_j by one unit, then the maximized value of your objective function will go up by λ_j units.
- Example: think of maximizing utility on a budget constraint. the higher are prices the longer is the gradient vector, and so the shorter is λ, i.e., the less bang for the buck you get, literally. I.e., relaxing your budget constraint by a dollar doesn't by much more of a bundle, so that your utility can't go up by much. On the other hand, for a fixed price vector, the length of the gradient of your utility function is a measure of how easy to please you are. If you are a kid who gets a lot of pleasure out of penny candy, then relaxing the budget constraint by a dollar will buy you a lot.
- the importance of this in economic applications is that you often want to know what the economic cost of a constraint is, e.g., suppose you are maximizing output subject to a resource constraint: what's it worth to you to increase the level of available resources by one unit. Get answer by looking at the Lagrangian.
- The mathematical proof is as usual completely trivial. Let $M(\mathbf{b})$ denote the maximized value of the objective function when the constraint vector is $\mathbf{b}$.

$$M(\mathbf{b}) \quad = \quad f(\bar{\mathbf{x}}(\mathbf{b})) \quad = \quad L(\bar{\mathbf{x}}(\mathbf{b}), \bar{\boldsymbol{\lambda}}(\mathbf{b})) \quad = \quad f(\mathbf{x}) + \boldsymbol{\lambda}(\mathbf{b} - \mathbf{g}(\mathbf{x})).$$

we have

$$\begin{aligned}\frac{dM(\mathbf{b})}{db_j} = \frac{dL(\bar{\mathbf{x}}(\mathbf{b}), \bar{\boldsymbol{\lambda}}(\mathbf{b}))}{db_j} \quad &= \quad \sum_{i=1}^{n} \left\{ f_i(\bar{\mathbf{x}}(\mathbf{b})) - \sum_{k=1}^{m} \lambda_k g_i^k(\bar{\mathbf{x}}(\mathbf{b})) \right\} \frac{dx_i}{db_j} \\ &\quad + \sum_{k=1}^{m} \frac{d\lambda_k}{db_j}(b_k - g^k(\mathbf{x})) + \lambda_j\end{aligned}$$

  - For each i, the term in curly brackets is zero, by the KT conditions.
  - For each k,

    * If $(b_k - g^k(\mathbf{x})) = 0$, then $\frac{d\lambda_k}{db_k}(b_k - g^k(\mathbf{x}))$ is zero.
    * If $(b_k - g^k(\mathbf{x})) < 0$, then $\lambda_k(\cdot)$ will be zero on a neighborhood of $\mathbf{b}$, so that $\frac{d\lambda_k(\cdot)}{db_k} = 0$.
  - The only term remaining is λ_j.
  - Conclude that $\frac{dM(\mathbf{b})}{db_j} = \lambda_j$
- Here's a more intuitive approach for the case of a single binding constraint. (Please refer back to the lecture notes for the Graphical Overview topic, esp mathGraphical3, for a detailed discussion of what binding means.)
  - increase the constraint b to $b + db$.
  - you should respond by moving from solution $\mathbf{x}$ in the direction of steepest ascent, i.e., in the direction that the gradient is pointing.
  - move from $\mathbf{x}$ in the direction that $\bigtriangledown f(\mathbf{x})$ is pointing until you reach the new constraint, i.e., define $\mathbf{dx}$ by $g(\mathbf{x} + \mathbf{dx}) = b + db$. Now initially we had $g(\mathbf{x}) = b$. Hence

$$db = g(\mathbf{x} + \mathbf{dx}) - g(\mathbf{x}) \approx \bigtriangledown g(\mathbf{x})\mathbf{dx}$$

    while

$$df = \bigtriangledown f(\mathbf{x})\mathbf{dx}$$

    which by the KT conditions

$$= \lambda \bigtriangledown g(\mathbf{x})\mathbf{dx}$$

    Hence

$$df = \lambda db$$

    Hence (being a little fast and loose with infinitesimals), $df/db = \lambda$.

Note that λ will be larger

- the more rapidly f increases with $\mathbf{x}$ (i.e., the longer is the vector $\bigtriangledown f(\mathbf{x})$).
- the *less* rapidly g increases with $\mathbf{x}$ (i.e., the *shorter* is the vector $\bigtriangledown g(\mathbf{x})$).

Note also that the above "proves" the 2nd part of the mantra, i.e., if constraint j is satisfied with equality but is not binding, then the weight on this constraint must be zero when you write $\bigtriangledown f(\mathbf{x})$ as a nonnegative linear combination of the $\bigtriangledown g^j(\mathbf{x})$'s. The following argument proves this rather sloppily: (a) if constraint j is satisfied with equality but is not binding, then by definition $\frac{df}{db_j} = 0$; (b) since $\frac{df}{db_j} = \lambda_j$, then λ_j must be zero.

6.6. **A worked solution to an NPP: S&B #18.18 (on the problem set)**

The example: S&B qu 18.18

$$\min 2x^2 + 2y^2 - 2xy - 9y \quad \text{s.t.}$$

$$\begin{aligned} 4x + 3y &\leq 10 \\ y - 4x^2 &\geq -2 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

Flip the signs so it's a max problem, respecify nonnegativity constraints as inequalities

$$\max f(x, y) = 2xy + 9y - 2x^2 - 2y^2 \quad \text{s.t.}$$

$$\begin{aligned} 4x + 3y &\leq 10 \\ 4x^2 - y &\leq 2 \\ -x &\leq 0 \\ -y &\leq 0 \end{aligned}$$

Now set up Lagrangian:

$$L(x, y, \boldsymbol{\lambda}) = (2xy + 9y - 2x^2 - 2y^2) + \lambda_1(10 - 4x - 3y) + \lambda_2(2 - 4x^2 + y) + \lambda_3 x + \lambda_4 y$$

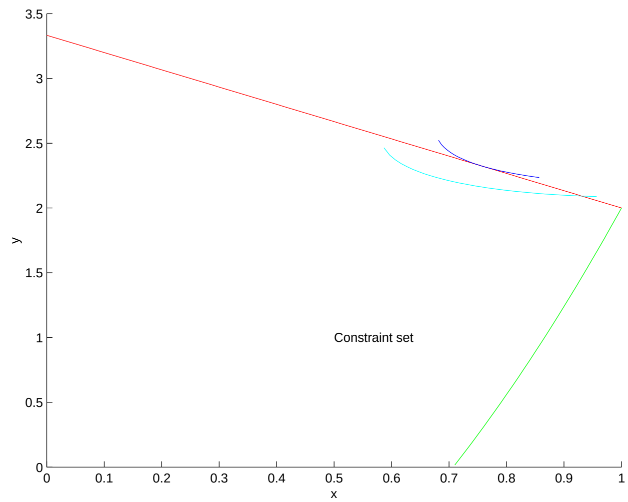

FIGURE 1. The feasible set for problem 18.18

Recall from (1), that the first order conditions were

$$\partial L(\bar{\mathbf{x}}, \bar{\boldsymbol{\lambda}})/\partial x_i = 0; \;\; \partial L(\bar{\mathbf{x}}, \bar{\boldsymbol{\lambda}})/\partial \lambda_j \geq 0; \;\; \bar{\lambda}_j \partial L(\bar{\mathbf{x}}, \bar{\boldsymbol{\lambda}})/\partial \lambda_j = 0. \tag{4}$$

In this particular problem, these conditions imply

$$\begin{aligned}
L_x &= 2y - 4x \quad - 4\lambda_1 - 8x\lambda_2 + \lambda_3 = 0 \\
L_y &= 2x - 4y + 9 - 3\lambda_1 + \lambda_2 + \lambda_4 = 0 \\
L_{\lambda_1} &= 10 - 4x - 3y \geq 0 \\
L_{\lambda_2} &= 2 - 4x^2 + y \geq 0 \\
L_{\lambda_3} &= x \geq 0 \\
L_{\lambda_4} &= y \geq 0
\end{aligned}$$

2

3) Assume at least one nonnegativity constraint satisfied with equality:

   (a) *First assume we're at the origin.* $x = y = 0$;

   - $L_{\lambda_1} > 0$; $L_{\lambda_2} > 0$
   - $\lambda_1 = \lambda_2 = 0$
   - $L_x = 0 - 0 - 0 - 0 + \lambda_3 = 0 \Longrightarrow \lambda_3 = 0$;
   - $L_y = 0 - 0 + 9 - 0 + 0 + \lambda_4 > 0.$ $\otimes$

   (b) *Now assume just x is positive* $x > y = 0$;

   - $\lambda_3 = 0$;
   - $L_{\lambda_2} = 2 - 4x^2 \geq 0 \Longrightarrow x \leq \sqrt{0.5}$
   - $L_{\lambda_3} > 10 - 4\sqrt{0.5} > 0 \Longrightarrow \lambda_1 = 0$
   - $L_y = 2x + 9 + \lambda_2 + \lambda_4 > 0.$ $\otimes$

   (c) *Now assume just y is positive* $y > x = 0$;

   - $\lambda_4 = 0$;
   - $L_{\lambda_2} = 2 + y > 0$
   - $\lambda_2 = 0$;
   - $L_x = 2y + \lambda_3 - 4\lambda_1 = 0 \Longrightarrow \lambda_1 > 0 \Longrightarrow y = 10/3$;
   - $L_y = -40/3 + 9 - 3\lambda_1 < 0$; $\otimes$

4) Conclude that $x > 0$; $y > 0$; $\lambda_3 = \lambda_4 = 0$.

5) Assume both x and y are positive:

   (a) $L_{\lambda_1} = L_{\lambda_2} = 0$

   - $\lambda_3 = \lambda_4 = 0$ (because both x and y are positive).
   - $L_{\lambda_1} = 10 - 4x - 3y = 0 \Longrightarrow y = (10 - 4x)/3$
   - $L_{\lambda_2} = 2 - 4x^2 + (10 - 4x)/3 = 0 \Longrightarrow 3x^2 + x - 4 = 0.$
   - i.e., $(3x + 4) * (x - 1) = 0 \Longrightarrow x = 1 \Longrightarrow y = 2.$
   - $L_y = 4 - 8 + 9 > 0$; $\otimes$

   (b) $L_{\lambda_1} > 0; L_{\lambda_2} = 0$ (only the quadratic constraint satisfied with equality)

   - $\lambda_1 = \lambda_3 = \lambda_4 = 0$
   - $L_{\lambda_2} = 2 - 4x^2 + y = 0 \Longrightarrow y = 4x^2 - 2$;
   - $L_x = 0 \Longrightarrow y \geq 2x$ (otherwise $L_x < 0$).

- $y = 4x^2 - 2$ and $y \geq 2x \Longrightarrow x \geq 1 \Longrightarrow y \geq 2$.
- $x \geq 1, y \geq 2 \Longrightarrow L_{\lambda_2} = 10 - 4x - 3y \leq 0$; $\otimes$

(c) $L_{\lambda_1} > 0; L_{\lambda_2} > 0$ (solution in the interior of constraint set)

- $\boldsymbol{\lambda} = 0$; (i.e. the whole vector zero)
- $L_x = 0 \Longrightarrow y = 2x$;
- $L_y = 0 \Longrightarrow 6x = 9 \Longrightarrow x = 1.5$;
- $y = 3$;
- $L_{\lambda_1} = 10 - 4 * 1.5 - 3 * 3 = -5$; $\otimes$

6) Computing the solution: $L_{\lambda_1} = 0; L_{\lambda_2} > 0$ (only the linear constraint satisfied with equality)

- $\lambda_2 = \lambda_3 = \lambda_4 = 0$
- $L_{\lambda_1} = 10 - 4x - 3y = 0 \Longrightarrow y = (10 - 4x)/3$
- $L_x = 2(10 - 4x)/3 - 4x - 4\lambda_1 = 0$ or $20 - 20x - 12\lambda_1 = 0$
- $L_y = 2x - 4(10 - 4x)/3 + 9 - 3\lambda_1 = 0$ or $22x - 13 - 9\lambda_1 = 0$

Two equations in two unknowns, i.e., $\begin{bmatrix} 20 & 12 \\ 22 & -9 \end{bmatrix} \begin{bmatrix} x \\ \lambda_1 \end{bmatrix} = \begin{bmatrix} 20 \\ 13 \end{bmatrix}$ so $x = 0.7568$; $\lambda_1 = 0.4054$;

Check that we have the answer using the mantra:

$$\begin{aligned} \bigtriangledown f &= [2y - 4x \quad 2x - 4y + 9] \\ &= [1.6216 \quad 1.2162] \propto \quad [4 \quad 3] \bigtriangledown g \quad = \quad [4 \quad 3] \end{aligned}$$

6.7. **Computing a solution to a NPP: a simple worked example**

How do you actually solve an NPP? Answer is: a process of elimination. You check all the possibilities to see if you can find a point that satisfies the KT conditions, and then you eliminate anything that fails this test. Here you are using the fact that the KT conditions are *necessary* for a solution, i.e., if they fail this test, they *can't* be a maximum. Once you've found something that does satisfy the KT conditions, then you have to go back and check that the second order conditions are satisfied.

The example:

$$\begin{aligned} \max f(\mathbf{x}) &= (x_1+2)(x_2-2) \text{ s.t.} \\ p_1x_1 + p_2x_2 &\leq y; \\ x_i &\geq 0; \end{aligned}$$

In this case, g is the matrix above, i.e.,

$$\begin{bmatrix} p_1 & p_2 \\ -1 & 0 \\ 0 & -1 \end{bmatrix};$$

Check the nonvanishing gradient condition: $\bigtriangledown f(\mathbf{x}) = 0$ iff $x_1 = -2$, $x_2 = 2$. Clearly this point is outside the constraint set, so we know the gradient is nonvanishing on the constraint set.

Set up Lagrangian:

$$L(\mathbf{x}, \lambda) = (x_1+2)(x_2-2) + \lambda_0(y - p_1x_1 - p_2x_2) + \lambda_1x_1 + \lambda_2x_2$$

Recall from (1), that the first order conditions were

$$\partial L(\bar{\mathbf{x}}, \bar{\boldsymbol{\lambda}})/\partial x_i = 0; \;\; \partial L(\bar{\mathbf{x}}, \bar{\boldsymbol{\lambda}})/\partial \lambda_j \geq 0; \;\; \bar{\lambda}_j \partial L(\bar{\mathbf{x}}, \bar{\boldsymbol{\lambda}})/\partial \lambda_j = 0.$$

In this particular problem, these conditions imply

$$\begin{aligned} L_{x_1} &= x_2 - 2 - \lambda_0 p_1 + \lambda_1 = 0 \\ L_{x_2} &= x_1 + 2 - \lambda_0 p_2 + \lambda_2 = 0 \\ L_{\lambda_0} &= y - p_1x_1 - p_2x_2 \geq 0 \\ L_{\lambda_1} &= x_1 \geq 0 \\ L_{\lambda_2} &= x_2 \geq 0. \end{aligned}$$

Observe that the last three FOC give you back precisely the constraint conditions.

We will set $p_1 = p_2 = y$ and solve explicitly for a solution. Under this condition, the solution will be at a corner. *NOTE WELL: this solution depends on the particular specification of parameters. In general, you could get a solution on the face of the budget line.*

Go through the interior, faces and vertices of the constraint set in turn. (Emphasize that while I can tell by inspection the solution to this problem, so I don't have to go thru all this hassle, in general I don't know the answer in advance, so don't have a clue about which corner, face, etc. to start with.)

(1) Try none of the constraints binding; KT says $\bar{\lambda}_0 = \bar{\lambda}_1 = \bar{\lambda}_2 = 0$. which implies $x = (-2, 2)$. Contradiction. Assumed that $\bar{x}$ was nonnegative; found that if there were a $\bar{x}$ that satisfied the KT conditions under these assumptions, then $\bar{x}_1$ would be negative. Note that we couldn't have had a point satisfying this condition anyway, because of the non-vanishing gradient property, we checked above

(2) Try $\bar{x}_1 > 0$, $\bar{x}_2 > 0$ and $p \cdot \bar{\mathbf{x}} = y$. KT conditions say that $\bar{\lambda}_1 = \bar{\lambda}_2 = 0$; $\bar{\lambda}_0 \geq 0$. Plugging these is gives

$$\begin{aligned} x_2 - 2 &= \lambda_0 p_1 \\ x_1 + 2 &= \lambda_0 p_2 \end{aligned}$$

But when $p_1 = p_2$, this means that $x_2 - x_1 = 4$. On the other hand, since $p_1 = p_2 = y$, the budget constraint implies that $x_1 + x_2 = 1$. Substituting yields $2x_2 = 5$, which implies x_1 MUST be negative, contradicting our initial condition that x_1 must be nonnegative.

(3) Try $\bar{x}_1 > 0$, $\bar{x}_2 = 0$ and $p \cdot \bar{\mathbf{x}} = y$. KT conditions say that $\bar{\lambda}_0, \bar{\lambda}_2 \geq 0$; $\bar{\lambda}_1 = 0$. Plugging these is gives

$$\begin{aligned} L_{x_1} &= x_2 - 2 - \lambda_0 p_1 + \lambda_1 \\ &= -2 - \lambda_0 p_1 = 0; \end{aligned}$$

which implies $\lambda_0 = -2/p_1$, which is a contradiction.

(4) Try $\bar{x}_1 = 0$, $\bar{x}_2 > 0$ and $p \cdot \bar{\mathbf{x}} = y$. KT conditions say that $\bar{\lambda}_0, \bar{\lambda}_1 \geq 0$; $\bar{\lambda}_2 = 0$. Plugging these in gives

$$L_{\lambda_0} = y - p_2 x_2 = 0$$

which implies $x_2 = y/p_2 > 0$. Also,

$$\begin{aligned} L_{x_2} &= x_1 + 2 - \lambda_0 p_2 \\ &= 2 - \lambda_0 p_2 = 0 \end{aligned}$$

which implies $\lambda_0 = 2/p_2$. Now consider L_{x_1}, i.e.,

$$\begin{aligned} L_{x_1} &= x_2 - 2 - \lambda_0 p_1 + \lambda_1 \\ &= y/p_2 - 2 - 2 + \lambda_1 \text{ (since } \lambda_0 = 2/p_1) \\ &= 1 - 4 + \lambda_1 \end{aligned}$$

which implies that $\lambda_1 = 3$. So we have a solution, i.e., $(0, y/p_2)$ with $\lambda_0 = 2/p_2$, $\lambda_1 = 3$.

6.8. **Computed solution to a NPP: ARE problem set example.**

This example was a homework problem for Econ 201A, 1999:

The example:

$$\begin{aligned} \max u_i(\mathbf{x}_i) &= x_{1i}(4 - x_{2i}) \text{ s.t.} \\ p_1 x_{1i} + p_2 x_{2i} &= p_1 \omega_{1i} + p_2 \omega_{2i}; \\ x_i &\geq 0; \end{aligned}$$

where $\omega_1 = (4, 3)$, $\omega_2 = (1, 0)$. The problem that 201 students faced was to solve for the $\mathbf{x}_i$'s for $i = 1, 2$, and for the equilibrium prices. What I'll do in these notes is to solve for the demand functions for good #1, *and* to derive some equilibrium properties of the price vector.

For this problem, g is the matrix above, i.e.,

$$\begin{bmatrix} p_1 & p_2 \\ -1 & 0 \\ 0 & -1 \end{bmatrix};$$

while b is

$$\begin{bmatrix} p_1\omega_{1i} + p_2\omega_{2i} \\ 0 \\ 0 \end{bmatrix};$$

We'll normalize by setting $p_1 = 1$ and let $\xi^i : \mathbb{R}^2 \to \mathbb{R}^2$ denote the demand function for agent i, i.e., the demand function is now a function only of relative prices. We know from the answer sheet that

$$\xi^1(p_2) = \begin{cases} \left(\frac{4-p_2}{2}, \frac{4+7p_2}{2p_2}\right) & \text{if } p_2 \leq -4/7 \\ (4+3p_2, 0) & \text{otherwise} \end{cases}$$

$$\xi^2(p_2) = \begin{cases} \left(\frac{1-4p_2}{2}, \frac{1+4p_2}{2p_2}\right) & \text{if } p_2 \leq -1/4 \\ (1, 0) & \text{otherwise} \end{cases}$$

We'll now derive the demand functions for agent #1, and check that our answers agree with ξ^1. You should check as an exercise that you can repeat the same steps for agent #2, and arrive at ξ^2.

Before embarking on this problem, we'll get some intuition for the solution. Note first from the definition of the utility function that

(1) provided that a positive quantity of good 1 is consumed, good 2 is a "bad";
(2) as x_{2i} increases above 4, then good 1 becomes a bad also, and the gradient of utility is a negative vector.

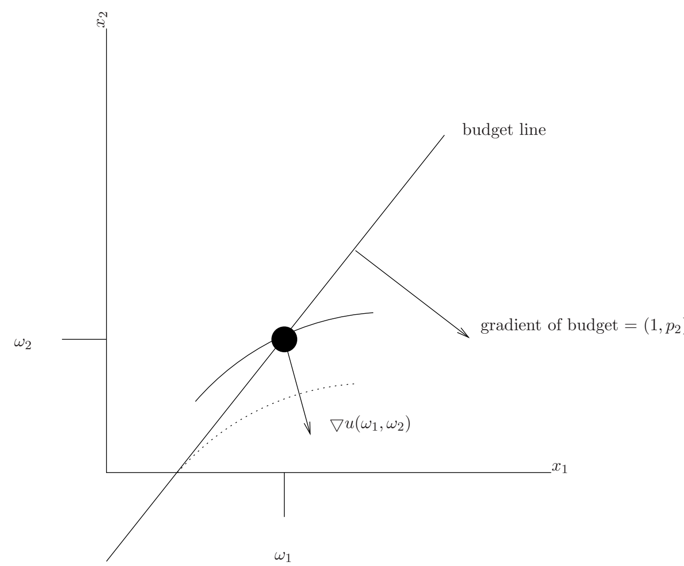

FIGURE 2. The problem facing agent #1

(3) it may not be *immediately* obvious, but when $x_{2i} > 4$, $u_i(\cdot)$ is a strictly quasi-*convex* function.

(4) now assume that both prices are both positive:

   (a) if $x_{2i} < 4$, you cannot get an interior (i.e., strictly positive) solution to the KT conditions because the gradient of the constraint is a strictly positive vector while the gradient of the utility function has one positive and one negative component.

   (b) if $x_{2i} > 4$, you *can* get an interior solution to the KT conditions because the graident of the utility function is strictly negative. In this solution, the *non-typical* constraint will be binding, i.e., instead of wanting to move NE in the positive quadrant you want to move SW. *However*, in this case the utility function is quasi-convex not quasi-concave. When you solve for an interior solution to the KT conditions, you'll have found a *minimum* on the constraint set, not a maximum.

(5) conclude from this that you cannot obtain an interior maximum to this problem if both prices are positive.

Rather than exploring all the possibilities exhaustively, we'll henceforth assume that $p_2 < 0$, i.e., good 2 is a bad. Now we'll draw the picture. The gradient of the budget constraint points down and to the right, i.e., SE., and the budget line is a positively sloped line through endowment point. Fig. 2 indicates the budget line with a relatively small negative price p_2; The optimum for this player is obviously a corner solution. Clearly, in order to get an interior solution to #1's optimization problem, you have to flatten the budget line, i.e., *lower* p_2.

From now on, I'm going to dump all the i subscripts since we're only dealing with $i = 1$.

Set up the Lagrangian, setting $p_1 = 1$ and $y(p_2) = \omega_1 + p_2\omega_2$, i.e., $y_1(p_2) = 4 + 3p_2$ and $y_2(p_2) = 1$.

$$L(\mathbf{x}, \lambda) = x_1(4 - x_2) + \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3(y(p_2) - x_1 - p_2 x_2) + \lambda_4(x_1 + p_2 x_2 - y(p_2))$$

Recall from (1), that the first order conditions were

$$\partial L(\bar{\mathbf{x}}, \bar{\boldsymbol{\lambda}})/\partial x_i = 0; \;\; \partial L(\bar{\mathbf{x}}, \bar{\boldsymbol{\lambda}})/\partial \lambda_j \geq 0; \;\; \bar{\lambda}_j \partial L(\bar{\mathbf{x}}, \bar{\boldsymbol{\lambda}})/\partial \lambda_j = 0. \tag{1}$$

In this particular problem, these conditions imply

$$\begin{aligned}
L_{x_1} &= (4 - x_2) + \lambda_1 - (\lambda_3 - \lambda_4) = 0 \\
L_{x_2} &= -x_1 + \lambda_2 - (\lambda_3 - \lambda_4)p_2 = 0 \\
L_{\lambda_1} &= x_1 \geq 0 \\
L_{\lambda_2} &= x_2 \geq 0 \\
L_{\lambda_3} &= y(p_2) - (x_1 + p_2 x_2) \geq 0 \\
L_{\lambda_4} &= x_1 + p_2 x_2 - y(p_2) \geq 0
\end{aligned}$$

Note that $L_{\lambda_3} > 0$ implies $L_{\lambda_4} < 0$ while $L_{\lambda_4} > 0$ implies $L_{\lambda_3} < 0$. Conclude that $L_{\lambda_3} = L_{\lambda_4} = 0$ leaving open the possibility that *either* λ_3 *or* λ_4 could be positive. Accordingly, it will be convenient to define $\lambda_0 = (\lambda_3 - \lambda_4)$, which can be either positive negative or zero. Thus for agent 1:

$$\begin{aligned} L_{x_1} &= (4 - x_2) + \lambda_1 - \lambda_0 = 0 \\ L_{x_2} &= -x_1 + \lambda_2 - \lambda_0 p_2 = 0 \\ L_{\lambda_0} &= x_1 + p_2 x_2 - (4 + 3p_2) = 0 \\ L_{\lambda_1} &= x_1 \geq 0 \\ L_{\lambda_2} &= x_2 \geq 0 \end{aligned}$$

By inspection of the figure we can see that there are really two possibilities:

(A) $p_2 < 0$; the budget line alone is binding (if p_2 is large in abs value, i.e., budget line relatively flat). In this case,

$$\lambda_1 = \lambda_2 = 0, \lambda_0 = \lambda_3 > 0$$

(B) $p_2 < 0$; the budget line and the nonneg constraint on good 2 are both binding (if p_2 is small in abs value, i.e., budget line relatively steep). In this case,

$$\lambda_1 = 0, \lambda_2 > 0, \lambda_0 = \lambda_3 > 0$$

On the other hand, Fig. 2 suggests that there are several possibilities that we can *exclude* based on the Lagrangian conditions. We'll focus on one of them, just for practice, but there are *many* more that we won't check.

(C) $p_2 > 0$ and $x_i > 0$, $i = 1, 2$, i.e., the budget line is the only constraint satisfied with equality.

We'll begin with (C), write down the Lagrangian system and show that all of the requirements *cannot simultaneously* be satisfied. From the mantra, we know the reason: the gradient of the budget line and the gradient of the objective have to be co-linear, but they can't be, because the objective's gradient points NE, while the budget's gradient points SE. Our task now is to show this using the Lagrangian. The conditions are:

$$\begin{aligned} L_{x_1} &= (4 - x_2) - \lambda_0 = 0 \\ L_{x_2} &= -x_1 - \lambda_0 p_2 = 0 \\ L_{\lambda_0} &= x_1 + p_2 x_2 - (4 + 3p_2) = 0 \\ \lambda_1 &= 0; \qquad \lambda_2 = 0; \end{aligned}$$

From L_{x_1} we have that

From L_{x_2} and $p_2 > 0$, we have that

$$\lambda_0 = 4 - x_2 \lambda_0 = \qquad -x_1/p_2 < 0$$

so that, substituting into L_{x_1}

$$(4 - x_2) + \frac{x_1}{p_2} = 0, \tag{5}$$

or, since $x_1 > 0$,

$$(x_2 - 4) = \frac{x_1}{p_2} > 0$$

But from L_{λ_0}, we have that

$$x_1 + p_2 x_2 = 4 + 3p_2 > 4$$

or

$$4 - x_2 > \frac{x_1}{p_2} > 0 \tag{6}$$

But, obviously, (5) and (6) cannot simultaneously be satisfied, so we've established that the set of conditions listed in (C) cannot hold. Note, moreover, that we obtained the contradiction by showing that if $p_2 > 0$, then the combination of L_{x_1} and L_{x_2} would then be inconsistent with L_{λ_0}.

Now let's consider the possibilities which from the figure, we know *are* possible. We will take each of possibilities (A) and (B) in turn, and see their implications for the Lagrangian system;

(A) the budget line alone is binding:

$$\begin{aligned} L_{x_1} &= (4 - x_2) - \lambda_0 = 0 \\ L_{x_2} &= -x_1 - \lambda_0 p_2 = 0 \\ L_{\lambda_0} &= x_1 + p_2 x_2 - (4 + 3p_2) = 0 \end{aligned}$$

Solving this in the usual way:

$$\begin{aligned} (a) \qquad 0 &= (4 - x_2) + \frac{x_1}{p_2} \qquad \text{(from } L_{x_1}\text{)} \\ (b) \qquad 0 &= p_2(4 - x_2) + x_1 \qquad \text{(rearranging (a))} \\ &= x_1 - p_2 x_2 + 4p_2 \\ (c) \qquad (4 + 3p_2) &= x_1 + p_2 x_2 \qquad \text{(from } L_{x_2}\text{)} \\ (d) \qquad (4 + 3p_2) &= 2p_2 x_2 - 4p_2 \qquad \text{(subtracting (b) from (c))} \\ (e) \qquad x_2 &= \frac{4 + 7p_2}{2p_2} \qquad \text{(rearranging (d))} \\ (f) \qquad x_1 &= \frac{4 - p_2}{2} \qquad \text{(subst (3) into (c))} \end{aligned}$$

Note that $x_2 \geq 0$ iff $|p_2| \leq 4/7$. Summarizing, (e) and (f) give us agent #1's demand function for $p_2 \in (-\infty, -4/7]$, i.e.,

$$\xi^1(p_2) = (\frac{4 + 7p_2}{2p_2}, \frac{4 - p_2}{2})$$

(B) both budget line and nonneg constraint on 2 are binding:

$$\begin{aligned} L_{x_1} &= 4 - \lambda_0 = 0 \\ L_{x_2} &= -x_1 + \lambda_2 - \lambda_0 p_2 = 0 \\ L_{\lambda_0} &= x_1 - (4 + 3p_2) = 0 \end{aligned}$$

From $L_{x_1} = 0$, $\lambda_0 = 4$. From $L_{\lambda_{01}} = 0$, $x_1 = (4 + 3p_2)$. Plugging both values into $L_{x_2} = 0$,

$$\begin{aligned} L_{x_2} &= -(4 + 3p_2) + \lambda_2 - 4p_2 \\ &= -4 - 7p_2 + \lambda_2 = 0; \end{aligned}$$

Now L_{x_2} can be zero with $\lambda_2 \geq 0$ iff $-7p_2 - 4 \leq 0$, i.e., if $|p_2| \leq 4/7$. Therefore, we have now computed agent #1's demand function for $p_2 \in (-4/7, 0]$, i.e.,

$$\xi^1(p_2) = (4 + 3p_2, 0)$$