ARE211, Fall 2009

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Contents

3. Linear Algebra (cont)

3.15. Eigenvalues, eigenvectors and difference Equations

Tremendously important application of eigenvalues and eigenvectors relates to difference equations.

- Consider a system of linear homogeneous first-order difference equations of the form $\mathbf{x}_t = A\mathbf{x}_{t-1}$.
- Think of this as an infinite system of equations, one for each time period.
- Each initial vector generates a sequence of vectors $\{\mathbf{x}_0, \mathbf{x}_1 \ldots\}$ s.t. $t > 0, \mathbf{x}_t = A\mathbf{x}_{t-1}$: any such sequence is called a *solution* to the system
- what are the properties of these solutions: do they converge to some specific vector?
- A *steady state/equilibrium/stationary solution* to an equation system is a vector $\mathbf{x}$ satisfying $\mathbf{x} = A\mathbf{x}$.
- Familiar result is that given a system of difference equations of the form $\mathbf{x}_t = A\mathbf{x}_{t-1}$ such that the eigenvalues of A are all distinct, the system has a unique steady state at $\mathbf{0}$ iff the largest (in absolute value) eigenvalue of A is less than unity. Trace out what happens: start with a circle, then ask what happens to the first ellipse, etc.
- Ignore the issue of unit eigenvalues.

- On the other hand, note that if one of the eigenvalues is greater than one in absolute value, then every starting vector that *is not* an eigenvector, eventually ends up pointing in a direction close to the direction of the eigenvector corresponding to the largest eigenvalue.
- Note the difference between behavior of difference equation systems depending on whether the matrix is positive or negative definite: monotonic behavior in the former case, big oscillations in the second case.
- Now in the case of difference equations we typically have to deal with asymmetric matrices, which give rise to much more interesting dynamics. Take the twisting matrix above, changing it a little so its eigenvalues are less than unity $F = \begin{bmatrix} \frac{4}{5} & \frac{1}{5} \\ \frac{-1}{5} & \frac{4}{5} \end{bmatrix}$. Note that the determinant of this matrix is less than one, so that the circle will spiral inwards: draw what happens to any starting point on the unit circle.
- Now consider a linear nonhomogeneous difference equations of the form $\mathbf{x}_t = A\mathbf{x}_{t-1} + \mathbf{b}$. Much richer dynamics. First solve for a steady state: $\mathbf{x}^* = A\mathbf{x}^* + \mathbf{b}$, ie., $\mathbf{x}^* = (I - A)^{-1}\mathbf{b}$, *provided that* the matrix $(I - A)$ is nonsingular. Now transform the nonhomogeneous system into a homogeneous one by subtracting the steady state equation from the original to obtain $(\mathbf{x}_t - \mathbf{x}^*) = A(\mathbf{x}_{t-1} - \mathbf{x}^*)$ which is an homogeneous system.
- Corresponding result for nonhomogeous difference equations of the form $\mathbf{x}_t = A\mathbf{x}_{t-1} + \mathbf{b}$: if all of the eigenvalues of A are distinct, and $(I - A)$ has full rank, then the system has a unique steady state at $(I - A)^{-1}\mathbf{b}$, iff the largest (in absolute value) eigenvalue of A is less than unity.