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3. LINEAR ALGEBRA (CONT)

3.6. Vector Spaces

A key concept in linear algebra is called a *vector space*. What is it? It is a special kind of set of vectors, satisfying a particular property. The property is a bit abstract, so we'll work up to it:

Verbal Defn: a nonempty set of vectors V is called a *vector space* if it satisfies the following property: given any two vectors that belong to the set V , *every* linear combination of these vectors is also in the space.

One kind of vector space is a set of vectors represented by a line. Note that a line can (and should) be thought of as just another set of vectors. Think about all of the lines you can draw in \mathbb{R}^2 . Which lines are vector spaces, according to the above definition?

- verbal defn excludes any straight line that doesn't go thru the origin.

- verbal defn excludes any line that “curve.”
- verbal defn excludes any straight line thru the origin that “stops.”

What’s left is rays thru the origin. Every ray through the origin is a vector space. Mathematically, any ray through the origin is defined as the set of all scalar multiples of a given vector. It’s called a “one-dimensional vector space”, because it is “constructed” from a single vector. Note that neither the nonnegative or the positive cones defined by a single vector are vector spaces.

Turns out that there’s exactly two more vector spaces in \mathbb{R}^2 :

- The set consisting of the entire plane is a vector space, i.e., \mathbb{R}^2 . This is a two dimensional vector space.
- The set consisting of zero alone is a vector space. This is a zero dimensional vector space.

Math Defn: a nonempty set of vectors V is called a *vector space* if for all $x^1, x^2 \in V$, for all $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_1 x^1 + \alpha_2 x^2 \in V$.

See how this definition stacks up against our examples:

- look at a line that curves: take any vector in the *set* consisting of the points in the line. Let $\mathbf{v}^1, \mathbf{v}^2$ be any vectors in the line. Take zero times the first and 2 times the second. Is it in the set? No. Conclude that the curved line is not a vector space.
- look at a straight line that doesn’t pass through the origin. Let $\mathbf{v}^1, \mathbf{v}^2$ be any vectors in the line. Take zero times both. In the set? No. Conclude that a straight line that doesn’t pass through the origin is not a vector space.
- take any straight line thru the origin that “stops.” Let $\mathbf{v}^1, \mathbf{v}^2$ be any vectors in the line. Take zero times the first and a large enough multiple times the second. Is it in the set? No. Conclude that a line thru the origin that “stops” is not a vector space.

Now consider zero: satisfies the above definition. Similarly the whole plane. But an arbitrary positive cone wouldn't.

What are the vector spaces in \mathbb{R}^3 .

- The zero vector (with three components).
- Lines thru the origin
- Planes that contain the origin
- The whole space \mathbb{R}^3 itself
- There aren't any others.

A vector space that lives inside another vector space is called a *vector subspace* of the original vector space.

- The vector space consisting of zero alone is a vector subspace of *every* vector space
- Lines thru the origin are vector subspaces of \mathbb{R}^2 , \mathbb{R}^3 or $\mathbb{R}^{\text{whatever}}$, depending on the number of components of the vector that defines the line.
- A plane in 3D that contains the origin is a vector subspace of \mathbb{R}^3 .

3.7. Spanning, Dimension, Basis

Intuitively, a set of vectors in \mathbb{R}^n is said to *span* a vector space if you can build the entire vector space out of the original set of vectors. (We'll always assume that the set of vectors are all conformable, i.e., have the same number of elements.)

Defn: Given a vector space V , a set of vectors $\{\mathbf{v}^1, \mathbf{v}^2, \dots, \mathbf{v}^i, \dots, \mathbf{v}^n\}$ spans V if any element of V can be written as a linear combination of \mathbf{v}^i 's.

Note that a given set of vectors can span many different spaces, some of which are subsets of others.

Example: Consider a set of vectors $\{\mathbf{v}^1, \mathbf{v}^2\} \subset \mathbb{R}^2$. Assume they are linearly independent. The different vector spaces that this set spans are:

- $\{(0, 0)\}$
- any line through $(0,0)$ (not necessarily one including either $\{\mathbf{v}^1$ or $\mathbf{v}^2\}$).
- \mathbb{R}^2

Moreover *every* set of vectors spans *some* vector space. Proof: the vector space consisting of $\{\mathbf{0}\} \subset \mathbb{R}^n$ is always spanned by any given set of vectors in \mathbb{R}^n .

Defn: The *span* of a set of vectors $\{\mathbf{v}^1, \mathbf{v}^2, \dots, \mathbf{v}^n\}$ is the (unique) vector space that contains the set and is spanned by the elements of the set.

That is, *the span* of a set of vectors is the biggest vector space that the set spans: all other spaces that the set spans are subspaces of *the span* of the set.

More examples of vector spaces and the vectors that span them:

- Take a line through the origin in \mathbb{R}^2 . Call this vector space V .
 - What vectors span this vector space? Answer: *any element* of V except one spans it. Exception is the vector $(0, 0)$, i.e., *all* of the elements of V can be written as scalar multiples of any given element of V (except zero).
 - Vectors that belong to the same line through the origin are called *collinear*, i.e., \mathbf{v}^1 and \mathbf{v}^2 are collinear if there exists $\alpha \in \mathbb{R}$ such that $\mathbf{v}^2 = \alpha\mathbf{v}^1$.
- Take the vector space consisting of zero. What spans it? Ans: zero.
- Take the whole of \mathbb{R}^2 : what sets of vectors span it? Answer: any pair of vectors that *don't both belong to* the same ray through the origin.
- Take the horizontal plane in \mathbb{R}^3 . What sets of vectors span it? How many vectors do you need?

- Take an arbitrary plane in \mathbb{R}^3 that passes through the origin. Not quite so easy to see that it is a vector space, but it is.
- Note that if a set of vectors spans a space and you throw in any number of additional vectors (with the same number of components) then the augmented set still spans the space. For example, if $\{\mathbf{v}^1, \mathbf{v}^2\}$ spans V then $\{\mathbf{v}^1, \mathbf{v}^2, \dots, \mathbf{v}^n\}$ does too. Just take $v' \in V$ and write it as a linear combination of \mathbf{v}^1 and \mathbf{v}^2 plus zero times all the other vectors.

Since spanning sets of vectors may contain many vectors that aren't necessary to span the space, it is useful to identify spanning sets of vectors that don't contain any excess fat. A *minimal spanning set* for a vector space is a set of vectors that spans the space, but if you took any element of the set out of the set, then you would no longer span the space. More generally

- Any single nonzero element of a line is a minimal spanning set.
- For the horizontal plane in \mathbb{R}^2 , any pair of noncolinear vectors in the plane spans it.

Examples of spanning sets that aren't minimal:

- Take a line through the origin, and a bunch of vectors belonging to the line, i.e., a bunch of collinear vectors. Can toss all but one out.
- Same thing for a plane in \mathbb{R}^3 : can toss all but two out.

Defn: Given a vector space V , a set of vectors $\{\mathbf{v}^1, \mathbf{v}^2, \dots, \mathbf{v}^n\} \subset V$ is a *basis* for V if it is a minimal spanning set for V , i.e., if it spans V and if any element of the set were omitted it would no longer span V .

Note the difference between a minimal spanning set and a basis: a basis for V is required to belong to V . Take $\{\mathbf{v}^1, \mathbf{v}^2\} \subset \mathbb{R}^2$ which are linearly independent so this set spans \mathbb{R}^2 . It also spans the subspace $\{\alpha \mathbf{v}^3 : \alpha \in \mathbb{R}\}$, where $\mathbf{v}^3 \notin \{\mathbf{v}^1, \mathbf{v}^2\}$. But $\{\mathbf{v}^1, \mathbf{v}^2\}$ is not a basis for this subspace.

Fact: (pretty obvious): if a set of vectors forms a basis for some vector space, then the set must be a linear independent set. To see this, suppose that \mathbf{v}^3 can be written as a linear combination of \mathbf{v}^1 and \mathbf{v}^2 . Then don't need \mathbf{v}^3 in order to build V . Use instead the more primitive building blocks \mathbf{v}^1 and \mathbf{v}^2 .

Theorem (seems obvious but in fact is quite deep, i.e., nontrivial to prove): Any two bases for a given vector space must have the same number of elements.

Examples:

- Lines thru the origin
- Planes in \mathbb{R}^3

Because of the preceding theorem, it makes sense to talk about the size of a vector space: size is the number of elements in any basis.

Defn: The *dimension* of a vector space V is the number of elements in any basis for V .

- A line through the origin is a one-dimensional vector space (or a one-dimensional vector subspace of \mathbb{R}^2).
- A Plane in 3D is a two-dimensional subspace of \mathbb{R}^3 .
- The vector space consisting of zero alone is a zero dimensional vector space.

Note well that we are talking now of a rather different usage of the word “dimension” from the familiar usage. Used to talking about 3 dimensional vectors, etc. A 3 dimensional vector space is something totally different from a 3 dimensional vector. (Almost totally different). Could have a 3 dimensional vector space that is a subspace of \mathbb{R}^{17} : its bases would all consist of three 17-dimensional vectors.

Fact: (also pretty obvious): the dimension of the vector space spanned by a given set of m vectors cannot have a dimension higher than m .

Fact: Exactly one vector space in \mathbb{R}^n has a unique set of basis vectors. For every other vector space, there are infinitely many sets of basis vectors. The exception is the zero-dimensional vector space. For any other vector space, take a basis, multiply every element by $\alpha \in \mathbb{R}$ and you have another one.

So far, we've been talking about vector subspaces of \mathbb{R}^n . There are lots of other vector spaces. For example, the set of all *sequences in* \mathbb{R} , i.e., the set of all mappings from \mathbb{N} to \mathbb{R} is a vector space. We'll call this space \mathbb{R}^∞ . What's a basis for this space? Obviously, it has to be a subset of the space, i.e., each element in the basis has to be a sequence in \mathbb{R} itself. There are lots of such bases.

One is the set $\{e^k\}_{k=1}^\infty$, where *each* e^k is itself a sequence, defined by $e_r^k = \begin{cases} 1 & \text{if } k = r \\ 0 & \text{otherwise} \end{cases}$. To see

that this is indeed a basis for \mathbb{R}^∞ , take any sequence $\{x_n\}_{n=1}^\infty$ and observe that it can be written as $\sum_{k=1}^\infty x_k e^k$.

3.8. Matrices and Rank

A matrix is nothing other than a bunch of column vectors stacked beside each other, or a bunch of row vectors stacked on top of each other. All of the above stuff about vector spaces and the sets that span them can be restated in terms of matrices:

- Can we say that a *matrix* spans a vector space? In fact any matrix spans two vector spaces: the columns span one, and the rows span another. If the matrix is $n \times m$, then the rows span a subspace of \mathbb{R}^m i.e., we have n rows each with m elements) and the columns span a subspace of \mathbb{R}^n
- Don't talk about the dimension of a matrix (well you do, but here the word dimension means something different), but you do talk about its *rank*

- The *row rank* of a matrix is the dimension of the (unique) vector space which is the span of the *rows* of the matrix.
- The *column rank* of a matrix is the dimension of the (unique) vector space which is the span of the *columns* of the matrix.
- (Deep fact if not pure magic:) The dimension of the vector space spanned by the rows is the same as the dimension of the vector space spanned by the columns. Matrix could have 14 rows and 348 columns. The rows span some subspace of \mathbb{R}^{348} . The columns span some subspace of \mathbb{R}^{14} . The dimensions of these entirely different subspaces (e.g., might be 7) are the same.
- For this reason, we can talk about *the rank* of a matrix, which is equal to both the row and the column rank.
- The rank of a matrix cannot exceed the minimum of the number of rows and the number of columns
- A matrix is said to be of *full rank* if its rank is equal to the minimum of the number of rows and the number of columns, that is, if it is as large as it can be.
- Under what circumstances do the rows or the columns of a matrix form a basis for a vector space? For an $n \times m$ matrix, with more columns than rows, i.e., $m > n$, the rows may form a basis for a vector space, but the columns can't:
 - * The rows will form a basis for an n -dimensional subspace of \mathbb{R}^m if and only if the rows are linearly independent
 - * The columns can't form a basis, because there are m of them and any space that they span must be a subspace of \mathbb{R}^n : now you need at most n vectors to do this and we have some extra ones
- Take a square matrix, M , which is $n \times n$. There are vector subspaces R and C of \mathbb{R}^n such that R is the span of the rows of the matrix and C is the span of the columns. What property of the matrix tells me whether or not R and C have dimension n or less? Ans again: linear independence.

Suppose the matrix M forms a basis for some vector space. Which vector space will it be: Ans: it must be \mathbb{R}^n . Note that if R and C have dimension less than n , they will in

general be *different* vector spaces. Under what conditions will $R = C$? Ans: if the matrix is symmetric.