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Contents

Preliminary draft only: please check for final version

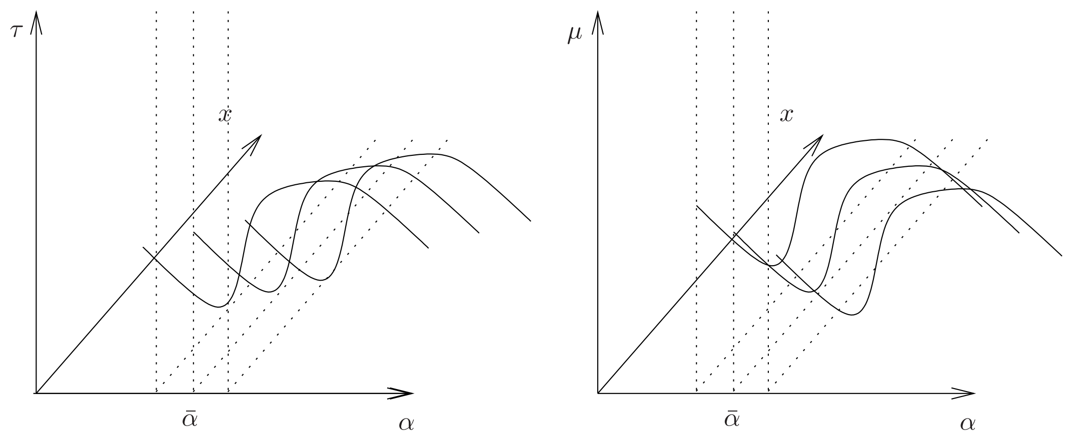
7. Implicit Function Theorem and the Envelope Theorem (cont)

7.4. Genericity and Transversality

Mas-Colell Whinston and Green have a difficult section on genericity, local uniqueness and the transversality theorem (pages 593-595). Fig. 1 and the following discussion is intended to shed some light on those pages.

Definition: (Definition 17.D.3. in MWG) a system of M equations in N unknowns $f(v)$ is *regular* if rank $Df(v) = M$ whenever $f(v) = 0$.

Fig. 1 graphs two functions τ and μ of α and x. The right panel is a 3-D version of MWG's figure 17.D.4. Note that if you fix α and consider μ as a function of x only (that is $M = N = 1$), then μ is not regular when $\alpha = \bar{\alpha}$, since there is a solution to $\mu(\bar{\alpha}, \cdot) = 0$ where the rank of the determinant is zero (i.e., at the point of tangency): in this case, $v = (\alpha, x)$ and $D\mu(v)$ is just $\mu_x = 0$. One way to interpret the transversality stuff is that, generally, there are other parameters in the system: if you interpret μ not as a system with one unknown but two (in our case, treat α as an unknown also), then provided the system varies with α in a way that's independent of the way it varies with x, then the existence of a non-regular equilibrium is highly unusual (a fluke in fact). In the right panel, note that as soon as you bump α a little bit, the freak tangency goes away. Given a function



Transversality theorem assumption fails

$\tau(\alpha, x) = (x-\alpha)^2(6+\alpha-x)$

Transversality theorem assumption satisfied

$\mu(\alpha, x) = (x-\alpha)^2(6+\alpha-x) + (1-\alpha)$

FIGURE 1. Local uniqueness and the transversality theorem

f, let $x_f(\alpha)$ be defined by the condition that $f(\alpha, x_f(\alpha)) = 0$. The right panel illustrates a function $\mu(\alpha, x)$ where $\mu_x(\bar{\alpha}, x_\mu(\bar{\alpha})) = 0$ but $\mu_\alpha(\bar{\alpha}, x_\mu(\bar{\alpha})) \neq 0$. That is $M = 1; N = 2$ and the rank of μ is M whenever $\mu(\alpha, x) = 0$, so the equation system is now regular. (That is, adding the extra variable "regularizes" the system.)

In the left panel, however, bumping α doesn't get rid of the tangency. In this case, the rank of τ at $\tau(\alpha, x(\alpha)) = 0$ is zero. (Specifically, the gradient τ is $(x-\alpha)$[something], and this is of course identically zero along the line $x = \alpha$.) So here we have an example of a non-regular system, whether it is considered a system with one or two unknowns.

The two panels also illustrate the transversality theorem, which I'll now state for the special case in which $M = 1$ and $N = 2$:

Theorem: (17.D.3 in MWG): if the $1 \times (1+1)$ matrix $Df(\alpha, x)$ has rank 1 whenever $f(\alpha, x) = 0$, then for almost every α, the 1×1 matrix $Df_x(x|\alpha)$ has full rank whenever $f(x|\alpha) = 0$.

In the left hand panel $D\tau(\alpha, x)$ has rank zero and for an open set of α's, $D\tau_x(x|\alpha)$ has less than full rank when $\tau(x|\alpha) = 0$. Thus, the condition of the theorem fails and the conclusion fails also. In the right hand panel $D\mu(\alpha, x)$ has rank one and with the exception of one value of α, i.e., $\bar{\alpha}$, $D\mu_x(x|\alpha)$ has rank one when $\mu(x|\alpha) = 0$. The condition of the theorem is satisfied and the conclusion is also.

The two panels of Fig. 1 highlight an interesting aspect of the implicit function theorem which we haven't discussed before. The right hand panel is exactly the "bad" case we discussed earlier, i.e., if you look at the level set $\{(\alpha, x) : \mu(\alpha, x) = 0\}$ in a nbd of $(\bar{\alpha}, x_\mu(\bar{\alpha}))$ you see that for α in every nbd to the left of $\bar{\alpha}$ there's no x value such that $\mu(\alpha, x) = 0$, and for α in every nbd to the right of $\bar{\alpha}$ there's two x values such that $\mu(\alpha, x) = 0$. On the other hand, in the left hand panel, which is the more pathological case from the point of view of the transversality theorem, there is a perfectly well defined, differentiable relationship $g(\cdot)$ between α and x, *even though* $\tau_x = 0$ *everywhere along the line* $(\alpha, g(\alpha)$*!*. So in this case, two bads (not only a tangency but a generic one) make a good, from the point of view of comparative statics theory. Of course none of this contradicts the implicit function theorem: it just points out that (a) the theorem provides sufficient but not necessary conditions for being able to find $g(\alpha)$, etc; (b) in this case, $g(\alpha)$ does exist, but you aren't going to find its' slope using the derivatives of the original function, f, whose level set defines g.