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Preliminary draft only: please check for final version

7. Implicit Function Theorem and the Envelope Theorem (cont)

7.5. Inverse function theorem

The inverse function theorem is a special case of the implicit function theorem. It applies to the case where $n = m$, i.e., same number of α's as x's and $\boldsymbol{f}(\boldsymbol{\alpha};\boldsymbol{x}) = \boldsymbol{\alpha} - \boldsymbol{\eta}(\boldsymbol{x})$, i.e., for each i, $f^i = \alpha_i - \eta^i(\boldsymbol{x})$ or $\alpha_i = \eta^i(\boldsymbol{x})$.

Theorem: Given $\boldsymbol{\eta} : \mathbb{R}^m \to \mathbb{R}^m$ and $\bar{\boldsymbol{\alpha}} \in \mathbb{R}^m$, If the determinant of $J\boldsymbol{\eta}(\bar{\boldsymbol{\alpha}})$ is not zero, then there exists a neighborhood of $\bar{\boldsymbol{\alpha}}$ and a differentiable function $\boldsymbol{\eta}^{-1} : \mathbb{R}^m \to \mathbb{R}^m$ such that on this neighborhood

$$\boldsymbol{\eta}^{-1}(\boldsymbol{x}) = \boldsymbol{x}$$

and

$$\begin{bmatrix} \frac{\partial(\eta^{-1})^1(\boldsymbol{x})}{\partial x_1} & \frac{\partial(\eta^{-1})^1(\boldsymbol{x})}{\partial x_2} & \cdots & \frac{\partial(\eta^{-1})^1(\boldsymbol{x})}{\partial x_m} \\ \frac{\partial(\eta^{-1})^2(\boldsymbol{x})}{\partial x_1} & \frac{\partial(\eta^{-1})^2(\boldsymbol{x})}{\partial x_2} & \cdots & \frac{\partial(\eta^{-1})^2(\boldsymbol{x})}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial(\eta^{-1})^m(\boldsymbol{x})}{\partial x_1} & \frac{\partial(\eta^{-1})^m(\boldsymbol{x})}{\partial x_2} & \cdots & \frac{\partial(\eta^{-1})^m(\boldsymbol{x})}{\partial x_m} \end{bmatrix} = \left(J\boldsymbol{\eta}(\bar{\boldsymbol{\alpha}})\right)^{-1}$$

We'll do it intuitively first, in one dimension. Suppose you have a function $\eta : \mathbb{R}^1 \rightarrow \mathbb{R}^1$ that is invertible, $(x = \eta(\alpha))$; i.e., every point in the range is associated with a unique point in the domain. In this case, you can certainly write α as a function of x; that is, define the function $\eta^{-1} : \mathbb{R}^1 \rightarrow \mathbb{R}^1$ such that $\eta^{-1}(x)$ picks out the α value that η took to x; mathematically, η^{-1} is defined by the condition that $\eta^{-1}(\eta(\alpha)) = \alpha$

- The closed form of the inverse may be very hard to compute, e.g., suppose $x = \exp^{\alpha} \times \sqrt{\sin(\alpha)/\cos(\alpha)}$. Could in principle define α as a function of x, but it could get messy.
- Easier to use the inverse function theorem, which says that $\partial\eta^{-1}(\cdot)/\partial x = (\partial\eta(\cdot)/\partial\alpha)^{-1}$.

Example: $x = \eta(\alpha) = 1/\alpha$, so that $\partial\eta(\cdot)/\partial\alpha = -1/\alpha^2$;

- first we'll take the derivative of the inverse by hand: we have
  - $\alpha = \eta^{-1}(x) = 1/x$,
  - $\partial\eta^{-1}(\cdot)/\partial x = -1/x^2$;
- now use the inverse function theorem:
  - $\partial\eta(\cdot)/\partial\alpha = -1/\alpha^2$;
  - applying the inverse function theorem $\partial\eta^{-1}(\cdot)/\partial x = (\partial\eta(\cdot)/\partial\alpha)^{-1} = (-1/\alpha^2)^{-1} = -\alpha^2$;
  - substitute x for α to obtain $\partial\eta^{-1}(\cdot)/\partial x = -1/x^2$;

Now return to the formalism of the implicit function theorem. When $m = n$, the expression from the last lecture becomes:

$$\begin{bmatrix} \frac{\partial g^1(\boldsymbol{\alpha})}{\partial\alpha_1} & \frac{\partial g^1(\boldsymbol{\alpha})}{\partial\alpha_2} & \cdots & \frac{\partial g^1(\boldsymbol{\alpha})}{\partial\alpha_n} \\ \frac{\partial g^2(\boldsymbol{\alpha})}{\partial\alpha_1} & \frac{\partial g^2(\boldsymbol{\alpha})}{\partial\alpha_2} & \cdots & \frac{\partial g^2(\boldsymbol{\alpha})}{\partial\alpha_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g^n(\boldsymbol{\alpha})}{\partial\alpha_1} & \frac{\partial g^n(\boldsymbol{\alpha})}{\partial\alpha_2} & \cdots & \frac{\partial g^n(\boldsymbol{\alpha})}{\partial\alpha_n} \end{bmatrix} = -\Gamma(\boldsymbol{\alpha},\boldsymbol{x})^{-1} \begin{bmatrix} \frac{\partial f^1(\boldsymbol{\alpha},\mathrm{g}(\boldsymbol{\alpha}))}{\partial\alpha_1} & \frac{\partial f^1(\boldsymbol{\alpha},\mathrm{g}(\boldsymbol{\alpha}))}{\partial\alpha_2} & \cdots & \frac{\partial f^1(\boldsymbol{\alpha},\mathrm{g}(\boldsymbol{\alpha}))}{\partial\alpha_n} \\ \frac{\partial f^2(\boldsymbol{\alpha},\mathrm{g}(\boldsymbol{\alpha}))}{\partial\alpha_1} & \frac{\partial f^2(\boldsymbol{\alpha},\mathrm{g}(\boldsymbol{\alpha}))}{\partial\alpha_2} & \cdots & \frac{\partial f^2(\boldsymbol{\alpha},\mathrm{g}(\boldsymbol{\alpha}))}{\partial\alpha_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f^n(\boldsymbol{\alpha},\mathrm{g}(\boldsymbol{\alpha}))}{\partial\alpha_1} & \frac{\partial f^n(\boldsymbol{\alpha},\mathrm{g}(\boldsymbol{\alpha}))}{\partial\alpha_2} & \cdots & \frac{\partial f^n(\boldsymbol{\alpha},\mathrm{g}(\boldsymbol{\alpha}))}{\partial\alpha_n} \end{bmatrix}.$$

where

$$\Gamma(\boldsymbol{\alpha},\boldsymbol{x}) \quad = \quad J^{\boldsymbol{x}}\boldsymbol{\eta}(\boldsymbol{\alpha},\boldsymbol{x}) \quad = \quad \begin{bmatrix} \frac{\partial f^1(\boldsymbol{\alpha};\boldsymbol{x})}{\partial x_1} & \frac{\partial f^1(\boldsymbol{\alpha};\boldsymbol{x})}{\partial x_2} & \cdots & \frac{\partial f^1(\boldsymbol{\alpha};\boldsymbol{x})}{\partial x_n} \\ \frac{\partial f^2(\boldsymbol{\alpha};\boldsymbol{x})}{\partial x_1} & \frac{\partial f^2(\boldsymbol{\alpha};\boldsymbol{x})}{\partial x_2} & \cdots & \frac{\partial f^2(\boldsymbol{\alpha};\boldsymbol{x})}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f^n(\boldsymbol{\alpha};\boldsymbol{x})}{\partial x_1} & \frac{\partial f^n(\boldsymbol{\alpha};\boldsymbol{x})}{\partial x_2} & \cdots & \frac{\partial f^n(\boldsymbol{\alpha};\boldsymbol{x})}{\partial x_n} \end{bmatrix}$$

When $m = n$, we can exchange $\boldsymbol{\alpha}$ and $\boldsymbol{x}$ as follows:

$$\begin{bmatrix} \frac{\partial g^1(\boldsymbol{x})}{\partial x_1} & \frac{\partial g^1(\boldsymbol{x})}{\partial x_2} & \cdots & \frac{\partial g^1(\boldsymbol{x})}{\partial x_n} \\ \frac{\partial g^2(\boldsymbol{x})}{\partial x_1} & \frac{\partial g^2(\boldsymbol{x})}{\partial x_2} & \cdots & \frac{\partial g^2(\boldsymbol{x})}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g^n(\boldsymbol{x})}{\partial x_1} & \frac{\partial g^n(\boldsymbol{x})}{\partial x_2} & \cdots & \frac{\partial g^n(\boldsymbol{x})}{\partial x_n} \end{bmatrix} = -\Gamma(\boldsymbol{x},\boldsymbol{\alpha})^{-1} \begin{bmatrix} \frac{\partial f^1(\boldsymbol{x},\mathrm{g}(\boldsymbol{x}))}{\partial x_1} & \frac{\partial f^1(\boldsymbol{x},\mathrm{g}(\boldsymbol{x}))}{\partial x_2} & \cdots & \frac{\partial f^1(\boldsymbol{x},\mathrm{g}(\boldsymbol{x}))}{\partial x_n} \\ \frac{\partial f^2(\boldsymbol{x},\mathrm{g}(\boldsymbol{x}))}{\partial x_1} & \frac{\partial f^2(\boldsymbol{x},\mathrm{g}(\boldsymbol{x}))}{\partial x_2} & \cdots & \frac{\partial f^2(\boldsymbol{x},\mathrm{g}(\boldsymbol{x}))}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f^n(\boldsymbol{x},\mathrm{g}(\boldsymbol{x}))}{\partial x_1} & \frac{\partial f^n(\boldsymbol{x},\mathrm{g}(\boldsymbol{x}))}{\partial x_2} & \cdots & \frac{\partial f^n(\boldsymbol{x},\mathrm{g}(\boldsymbol{x}))}{\partial x_n} \end{bmatrix}.$$

where

$$-\Gamma(\boldsymbol{x},\boldsymbol{\alpha}) = -J^{\boldsymbol{\alpha}}\boldsymbol{\eta}(\boldsymbol{x},\boldsymbol{\alpha}) = -\begin{bmatrix} \frac{\partial f^1(\boldsymbol{x};\boldsymbol{\alpha})}{\partial\alpha_1} & \frac{\partial f^1(\boldsymbol{x};\boldsymbol{\alpha})}{\partial\alpha_2} & \cdots & \frac{\partial f^1(\boldsymbol{x};\boldsymbol{\alpha})}{\partial\alpha_n} \\ \frac{\partial f^2(\boldsymbol{x};\boldsymbol{\alpha})}{\partial\alpha_1} & \frac{\partial f^2(\boldsymbol{x};\boldsymbol{\alpha})}{\partial\alpha_2} & \cdots & \frac{\partial f^2(\boldsymbol{x};\boldsymbol{\alpha})}{\partial\alpha_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f^n(\boldsymbol{x};\boldsymbol{\alpha})}{\partial\alpha_1} & \frac{\partial f^n(\boldsymbol{x};\boldsymbol{\alpha})}{\partial\alpha_2} & \cdots & \frac{\partial f^n(\boldsymbol{x};\boldsymbol{\alpha})}{\partial\alpha_n} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial \eta^1(\boldsymbol{x};\boldsymbol{\alpha})}{\partial\alpha_1} & \frac{\partial \eta^1(\boldsymbol{x};\boldsymbol{\alpha})}{\partial\alpha_2} & \cdots & \frac{\partial \eta^1(\boldsymbol{x};\boldsymbol{\alpha})}{\partial\alpha_n} \\ \frac{\partial \eta^2(\boldsymbol{x};\boldsymbol{\alpha})}{\partial\alpha_1} & \frac{\partial \eta^2(\boldsymbol{x};\boldsymbol{\alpha})}{\partial\alpha_2} & \cdots & \frac{\partial \eta^2(\boldsymbol{x};\boldsymbol{\alpha})}{\partial\alpha_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \eta^n(\boldsymbol{x};\boldsymbol{\alpha})}{\partial\alpha_1} & \frac{\partial \eta^n(\boldsymbol{x};\boldsymbol{\alpha})}{\partial\alpha_2} & \cdots & \frac{\partial \eta^n(\boldsymbol{x};\boldsymbol{\alpha})}{\partial\alpha_n} \end{bmatrix}$$

Notice how the two minus signs cancel each other out. Notice also that *all* we have done here is to switch the endog and the exog variables.

We're no longer going to worry about whether the function is *globally* invertible, and just talk about when it's *locally* invertible, i.e., when i.e., $\Gamma(\boldsymbol{x},\boldsymbol{\alpha})$ defined above is an invertible matrix. In the special case where $\boldsymbol{f}(\boldsymbol{x};\boldsymbol{\alpha}) = \boldsymbol{x} - \boldsymbol{\eta}(\boldsymbol{\alpha})$, the second matrix in equation (**??**) becomes the identity matrix and we have, locally, $\boldsymbol{\alpha} = \boldsymbol{g}(\boldsymbol{x})$, where

$$\begin{bmatrix} \frac{\partial g^1(\boldsymbol{x})}{\partial x_1} & \frac{\partial g^1(\boldsymbol{x})}{\partial x_2} & \cdots & \frac{\partial g^1(\boldsymbol{x})}{\partial x_n} \\ \frac{\partial g^2(\boldsymbol{x})}{\partial x_1} & \frac{\partial g^2(\boldsymbol{x})}{\partial x_2} & \cdots & \frac{\partial g^2(\boldsymbol{x})}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g^n(\boldsymbol{x})}{\partial x_1} & \frac{\partial g^n(\boldsymbol{x})}{\partial x_2} & \cdots & \frac{\partial g^n(\boldsymbol{x})}{\partial x_n} \end{bmatrix} = -\Gamma(\boldsymbol{x},\boldsymbol{\alpha})^{-1} = \Big(J^{\boldsymbol{\alpha}}\boldsymbol{\eta}(\boldsymbol{x},\boldsymbol{\alpha})\Big)^{-1} \tag{1}$$

When $m = n = 1$, this reduces, of course, to $\alpha = (\eta'(x))^{-1}$.

Here's an economic example of how the inverse and implicit function theorems might by combined: consider a joint production function $\boldsymbol{\zeta} : \boldsymbol{x} \to \boldsymbol{q}$, where inputs $\boldsymbol{x}$ and outputs $\boldsymbol{q}$ are elements of $\mathbb{R}^n$. First let's construct the *cost function* $C(\boldsymbol{q}) = \boldsymbol{w} \cdot \boldsymbol{x}(\boldsymbol{q})$, where $\boldsymbol{x}$ is chosen to minimize costs given input prices $\boldsymbol{w}$. That is, $\boldsymbol{x}(\boldsymbol{q})$ solves the NPP

$$\min_{\boldsymbol{x}} \boldsymbol{w} \cdot \boldsymbol{x} \quad \text{s.t.} \quad \boldsymbol{q} = \boldsymbol{\zeta}(\boldsymbol{x})$$

To solve this, set up the Lagrangian:

$$L(\boldsymbol{q},\boldsymbol{w};\boldsymbol{x},\boldsymbol{\lambda}) \quad = \quad -\boldsymbol{w}\cdot\boldsymbol{x} + \sum_{i=1}^{n}\lambda^i(q^i - \zeta^i(\boldsymbol{x}))$$

The first order conditions for L give us $\boldsymbol{x}(\boldsymbol{q})$. Moreover, $C(\boldsymbol{q}) = L(\boldsymbol{x}(\boldsymbol{q}),\boldsymbol{\lambda};\boldsymbol{w})$. Finally, the *marginal cost function* is given by $\mathrm{MC}(\boldsymbol{q}) = \left[\frac{\partial C(\boldsymbol{q})}{\partial q_1},\cdots,\frac{\partial C(\boldsymbol{q})}{\partial q_n}\right]$, and

$$\frac{\partial C(\boldsymbol{q})}{\partial q_i} \quad = \quad \sum_{j=1}^{n} w_j \frac{\partial x_j(\boldsymbol{q})}{\partial q_i}$$

where, applying the implicit function theorem

$$\begin{bmatrix} \frac{\partial x_1(\boldsymbol{q})}{\partial q_1} & \frac{\partial x_1(\boldsymbol{q})}{\partial q_2} & \cdots & \frac{\partial x_1(\boldsymbol{q})}{\partial q_n} \\ \frac{\partial x_2(\boldsymbol{q})}{\partial q_1} & \frac{\partial x_2(\boldsymbol{q})}{\partial q_2} & \cdots & \frac{\partial x_2(\boldsymbol{q})}{\partial q_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n(\boldsymbol{q})}{\partial q_1} & \frac{\partial x_n(\boldsymbol{q})}{\partial q_2} & \cdots & \frac{\partial x_n(\boldsymbol{q})}{\partial q_n} \end{bmatrix} \quad = \quad -\Big(\Gamma(\boldsymbol{q},\boldsymbol{w};\boldsymbol{x}(\boldsymbol{q}),\boldsymbol{\lambda}(\boldsymbol{q}))\Big)^{-1} \begin{bmatrix} \frac{\partial L_{x_1}(\boldsymbol{q},\boldsymbol{w};\boldsymbol{x}(\boldsymbol{q}),\boldsymbol{\lambda}(\boldsymbol{q}))}{\partial q_1} & \cdots & \frac{\partial L_{x_1}(\boldsymbol{q},\boldsymbol{w};\boldsymbol{x}(\boldsymbol{q}),\boldsymbol{\lambda}(\boldsymbol{q}))}{\partial q_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial L_{x_n}(\boldsymbol{q},\boldsymbol{w};\boldsymbol{x}(\boldsymbol{q}),\boldsymbol{\lambda}(\boldsymbol{q}))}{\partial q_1} & \cdots & \frac{\partial L_{x_n}(\boldsymbol{q},\boldsymbol{w};\boldsymbol{x}(\boldsymbol{q}),\boldsymbol{\lambda}(\boldsymbol{q}))}{\partial q_n} \\ \frac{\partial L_{\lambda_1}(\boldsymbol{q},\boldsymbol{w};\boldsymbol{x}(\boldsymbol{q}),\boldsymbol{\lambda}(\boldsymbol{q}))}{\partial q_1} & \cdots & \frac{\partial L_{\lambda_1}(\boldsymbol{q},\boldsymbol{w};\boldsymbol{x}(\boldsymbol{q}),\boldsymbol{\lambda}(\boldsymbol{q}))}{\partial q_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial L_{\lambda_n}(\boldsymbol{q},\boldsymbol{w};\boldsymbol{x}(\boldsymbol{q}),\boldsymbol{\lambda}(\boldsymbol{q}))}{\partial q_1} & \cdots & \frac{\partial L_{\lambda_n}(\boldsymbol{q},\boldsymbol{w};\boldsymbol{x}(\boldsymbol{q}),\boldsymbol{\lambda}(\boldsymbol{q}))}{\partial q_n} \end{bmatrix}.$$

and

$$\Gamma(\boldsymbol{q},\boldsymbol{w};\boldsymbol{x}(\boldsymbol{q}),\boldsymbol{\lambda}(\boldsymbol{q})) \quad = \quad \begin{bmatrix} \frac{\partial L_{x_1}(\boldsymbol{q},\boldsymbol{w};\boldsymbol{x}(\boldsymbol{q}),\boldsymbol{\lambda}(\boldsymbol{q}))}{\partial x_1} & \cdots & \frac{\partial L_{x_1}(\boldsymbol{q},\boldsymbol{w};\boldsymbol{x}(\boldsymbol{q}),\boldsymbol{\lambda}(\boldsymbol{q}))}{\partial x_n} & \frac{\partial L_{x_1}(\boldsymbol{q},\boldsymbol{w};\boldsymbol{x}(\boldsymbol{q}),\boldsymbol{\lambda}(\boldsymbol{q}))}{\partial \lambda_1} & \cdots & \frac{\partial L_{x_1}(\boldsymbol{q},\boldsymbol{w};\boldsymbol{x}(\boldsymbol{q}),\boldsymbol{\lambda}(\boldsymbol{q}))}{\partial \lambda_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial L_{x_n}(\boldsymbol{q},\boldsymbol{w};\boldsymbol{x}(\boldsymbol{q}),\boldsymbol{\lambda}(\boldsymbol{q}))}{\partial x_1} & \cdots & \frac{\partial L_{x_n}(\boldsymbol{q},\boldsymbol{w};\boldsymbol{x}(\boldsymbol{q}),\boldsymbol{\lambda}(\boldsymbol{q}))}{\partial x_n} & \frac{\partial L_{x_n}(\boldsymbol{q},\boldsymbol{w};\boldsymbol{x}(\boldsymbol{q}),\boldsymbol{\lambda}(\boldsymbol{q}))}{\partial \lambda_1} & \cdots & \frac{\partial L_{x_n}(\boldsymbol{q},\boldsymbol{w};\boldsymbol{x}(\boldsymbol{q}),\boldsymbol{\lambda}(\boldsymbol{q}))}{\partial \lambda_n} \\ \frac{\partial L_{\lambda_1}(\boldsymbol{q},\boldsymbol{w};\boldsymbol{x}(\boldsymbol{q}),\boldsymbol{\lambda}(\boldsymbol{q}))}{\partial x_1} & \cdots & \frac{\partial L_{\lambda_1}(\boldsymbol{q},\boldsymbol{w};\boldsymbol{x}(\boldsymbol{q}),\boldsymbol{\lambda}(\boldsymbol{q}))}{\partial x_n} & \frac{\partial L_{\lambda_1}(\boldsymbol{q},\boldsymbol{w};\boldsymbol{x}(\boldsymbol{q}),\boldsymbol{\lambda}(\boldsymbol{q}))}{\partial \lambda_1} & \cdots & \frac{\partial L_{\lambda_1}(\boldsymbol{q},\boldsymbol{w};\boldsymbol{x}(\boldsymbol{q}),\boldsymbol{\lambda}(\boldsymbol{q}))}{\partial \lambda_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial L_{\lambda_n}(\boldsymbol{q},\boldsymbol{w};\boldsymbol{x}(\boldsymbol{q}),\boldsymbol{\lambda}(\boldsymbol{q}))}{\partial x_1} & \cdots & \frac{\partial L_{\lambda_n}(\boldsymbol{q},\boldsymbol{w};\boldsymbol{x}(\boldsymbol{q}),\boldsymbol{\lambda}(\boldsymbol{q}))}{\partial x_n} & \frac{\partial L_{\lambda_n}(\boldsymbol{q},\boldsymbol{w};\boldsymbol{x}(\boldsymbol{q}),\boldsymbol{\lambda}(\boldsymbol{q}))}{\partial \lambda_1} & \cdots & \frac{\partial L_{\lambda_n}(\boldsymbol{q},\boldsymbol{w};\boldsymbol{x}(\boldsymbol{q}),\boldsymbol{\lambda}(\boldsymbol{q}))}{\partial \lambda_n} \end{bmatrix}$$

Now suppose the firm using $\boldsymbol{\zeta}$ is a competitive firm, so that it sets $\mathrm{MC}(\boldsymbol{q}) = \boldsymbol{p}$. Ultimately, we want to know how the vector of factors with $\boldsymbol{p}$. Set $\boldsymbol{h}(\boldsymbol{p};\boldsymbol{q}) = \boldsymbol{p} - \mathrm{MC}(\boldsymbol{q})$ and note that $\boldsymbol{h}$ is identically zero. I.e., $\boldsymbol{h}(\boldsymbol{p};\boldsymbol{q})$ is of the form $\boldsymbol{f}(\boldsymbol{\alpha};\boldsymbol{x})$. So here's a case in which the economics gives us $\boldsymbol{p}$ in terms of $\boldsymbol{q}$, i.e., the exogenous variable in terms of the endogenous variable. We of course want to know the reverse relationship, i.e., how $\boldsymbol{q}$ changes as prices change. To get this, we apply the *inverse* function theorem to obtain the matrix of $\frac{dq_j}{dp_i}$'s.

$$\begin{bmatrix} \frac{\partial q^1(\boldsymbol{x})}{\partial p_1} & \frac{\partial q^1(\boldsymbol{x})}{\partial p_2} & \cdots & \frac{\partial q^1(\boldsymbol{x})}{\partial p_n} \\ \frac{\partial q^2(\boldsymbol{x})}{\partial p_1} & \frac{\partial q^2(\boldsymbol{x})}{\partial p_2} & \cdots & \frac{\partial q^2(\boldsymbol{x})}{\partial p_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial q^n(\boldsymbol{x})}{\partial p_1} & \frac{\partial q^n(\boldsymbol{x})}{\partial p_2} & \cdots & \frac{\partial q^n(\boldsymbol{x})}{\partial p_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial MC^1(\boldsymbol{x})}{\partial q_1} & \frac{\partial MC^1(\boldsymbol{x})}{\partial q_2} & \cdots & \frac{\partial MC^1(\boldsymbol{x})}{\partial q_n} \\ \frac{\partial MC^2(\boldsymbol{x})}{\partial q_1} & \frac{\partial MC^2(\boldsymbol{x})}{\partial q_2} & \cdots & \frac{\partial MC^2(\boldsymbol{x})}{\partial q_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial MC^n(\boldsymbol{x})}{\partial q_1} & \frac{\partial MC^n(\boldsymbol{x})}{\partial q_2} & \cdots & \frac{\partial MC^n(\boldsymbol{x})}{\partial q_n} \end{bmatrix}^{-1}$$