ARE211, Fall 2009

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Contents

7. Foundations of Comparative Statics (cont)

7.3. ARE Prelim questions

7.3.1. *ARE201 (International Trade), 2007.* :

Question: Consider the problem

$$y(\alpha) \quad = \quad \max_x \frac{g(\alpha x)}{\alpha}$$

Let $x^*(\alpha)$ denote the solution to this problem

(1) Write the expression for the elasticity of x^* w.r.t. α

(2) Write the expression for $\frac{dy}{d\alpha}$.

Answer: Let $f(x,\alpha) = \frac{g(\alpha x)}{\alpha}$.

(1) By the implicit function theorem,

$$\frac{dx^*(\alpha)}{d\alpha} = -\frac{f_\alpha(x^*(\alpha),\alpha)}{f_x(x^*(\alpha),\alpha)}$$

Now

$$\begin{aligned} f_\alpha(x^*(\alpha),\alpha) &= x^*(\alpha)g'(\alpha x^*(\alpha))/\alpha - g(\alpha x^*(\alpha))/\alpha^2 \\ f_x(x^*(\alpha),\alpha) &= g'(\alpha x^*(\alpha)) \end{aligned}$$

Therefore

$$\frac{dx^*(\alpha)}{d\alpha} = -\big(x^*(\alpha)/\alpha - g(\alpha x^*(\alpha))/(\alpha^2 g'(\alpha x^*(\alpha)))\big)$$

and

$$\begin{aligned} \text{Elasticity} &= \frac{dx^*(\alpha)}{d\alpha}\frac{\alpha}{x^*(\alpha)} = -\left(\frac{x^*(\alpha)}{\alpha}\frac{\alpha}{x^*(\alpha)} - \frac{g(\alpha x^*(\alpha))}{\alpha^2 g'(\alpha x^*(\alpha))}\frac{\alpha}{x^*(\alpha)}\right) \\ &= \frac{g(\alpha x^*(\alpha))}{\alpha x^*(\alpha) g'(\alpha x^*(\alpha))} - 1 \end{aligned}$$

(2) By the envelope theorem

$$\begin{aligned} \frac{dy}{d\alpha} &= \frac{\partial y}{\partial \alpha} = \frac{\partial f(x^*(\alpha),\alpha)}{\partial \alpha} \\ &= x^*(\alpha)g'(\alpha x^*(\alpha))/\alpha - g(\alpha x^*(\alpha))/\alpha^2 \\ &= \frac{\alpha x^*(\alpha)g'(\alpha x^*(\alpha)) - g(\alpha x^*(\alpha))}{\alpha^2} \end{aligned}$$

Note that this answer would be incorrect had you just written the answer as $\frac{\alpha x g'(\alpha x) - g(\alpha x)}{\alpha^2}$

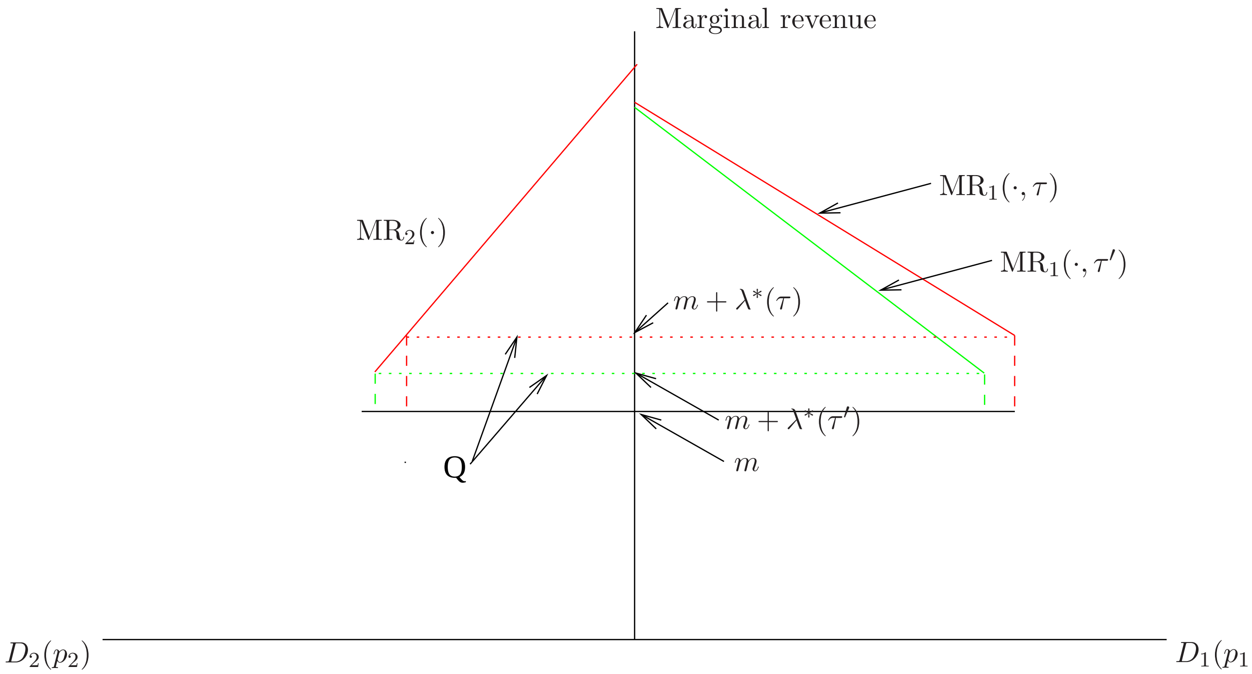

FIGURE 1. Monopolist in two markets with a quantity constraint

7.3.2. *Implicit function question: ARE prelim, 2005, #2.*

Question: A monopolist sells in two countries: 1 and 2. It produces a good at a constant marginal cost of m and cannot produce more than a total of Q units. Country 1 imposes a per unit tax of τ units on the good sold in that country. Assume that the constraint binds.

(1) For a given value of τ, show how the equilibrium prices in the two countries are determined.
(2) Show how the prices change as τ increases.

Answer: To make our lives easy, we'll assume that in both countries, the inverse demand curve is concave in price. We'll also assume that both quantities are positive in equilibrium, so we don't have to deal with corner solutions. We'll now write the monopolist's optimization problem as

$$\max_{p_1,p_2} \quad (p_1-\tau)D_1(p_1)+p_2D_2(p_2)-m(D_1(p_1)+D_2(p_2)) \quad \text{s.t.} \qquad D_1(p_1)+D_2(p_2)\le Q$$

$$L(p_1,p_2,\lambda;\tau) \quad = \quad (p_1-\tau)D_1(p_1)+p_2D_2(p_2)-m(D_1(p_1)+D_2(p_2))+\lambda\left(Q-D_1(p_1)-D_2(p_2)\right)$$

so, assuming the capacity constraint binds, the first order conditions are

$$\begin{aligned} L_{p_1} &= D_1(p_1) + (p_1 - \tau - m - \lambda)\, D_1'(p_1) = 0 \\ L_{p_2} &= D_2(p_2) + (p_2 \quad - m - \lambda)\, D_2'(p_2) = 0 \\ L_\lambda &= Q - D_1(p_1) - D_2(p_2) = 0 \end{aligned}$$

So the solution to this problem is a triple $(p_1^*, p_2^*, \lambda^*)$ which solves this equation system. To figure out how the tax impacts the solution, we apply the implicit function theorem. The Hessian of the Lagrangian is

$$\text{HL} = \begin{bmatrix} 2D_1'(p_1^*) + (p_1^* - \tau - m - \lambda^*)\, D_1''(p_1^*) & 0 & -D_1'(p_1^*) \\ 0 & 2D_2'(p_2^*) + (p_2^* - m - \lambda^*)\, D_2''(p_2^*) & -D_2'(p_2^*) \\ -D_1'(p_1^*) & -D_2'(p_2^*) & 0 \end{bmatrix}$$

while the derivative of the first order conditions w.r.t. τ is

$$\begin{bmatrix} -D_1'(p_1) \\ 0 \\ 0 \end{bmatrix}$$

hence from the implicit function theorem, we have

$$\begin{bmatrix} \frac{dp_1^*}{d\tau} \\ \frac{dp_2^*}{d\tau} \\ \frac{d\lambda}{d\tau} \end{bmatrix} = -\,\text{HL}^{-1} \begin{bmatrix} -D_1'(p_1) \\ 0 \\ 0 \end{bmatrix} = \text{HL}^{-1} \begin{bmatrix} D_1'(p_1) \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

It's straightforward to check that the determinant of HL is

$$-\left[D_2'(p_2^*)^2 \left\{2D_1'(p_1^*) + (p_1^* - \tau - m - \lambda^*)\, D_1''(p_1^*)\right\} + D_1'(p_1^*)^2 \left\{2D_2'(p_2^*) + (p_2^* - m - \lambda^*)\, D_2''(p_2^*)\right\}\right] > 0$$

Applying Cramer's rule, we have

$$\frac{dp_1^*}{d\tau} = \det\left(\begin{bmatrix} D_1'(p_1^*) & 0 & -D_1'(p_1^*) \\ 0 & 2D_2'(p_2^*) + \,(p_2^* - m - \lambda^*)\, D_2''(p_2^*) & -D_2'(p_2^*) \\ 0 & -D_2'(p_2^*) & 0 \end{bmatrix}\right) \Big/ \det(HL)$$

$$= -D_1'(p_1^*) D_2'(p_2^*)^2 \big/ \det(HL) > 0$$

$$\frac{dp_2^*}{d\tau} = \det\left(\begin{bmatrix} 2D_1'(p_1^*) + \,(p_1^* - m - \tau - \lambda^*)\, D_1''(p_1^*) & D_1'(p_1^*) & -D_1'(p_1^*) \\ 0 & 0 & -D_2'(p_1^*) \\ -D_1'(p_2^*) & 0 & 0 \end{bmatrix}\right) \Big/ \det(HL)$$

$$= D_2'(p_1^*) D_1'(p_2^*)^2 \big/ \det(HL) < 0$$

$$\frac{d\lambda^*}{d\tau} = \det\left(\begin{bmatrix} 2D_1'(p_1^*) + \,(p_1^* - \tau - m - \lambda^*)\, D_1''(p_1^*) & 0 & D_1'(p_1^*) \\ 0 & 2D_2'(p_2^*) + \,(p_2^* - m - \lambda^*)\, D_2''(p_2^*) & 0 \\ -D_1'(p_1^*) & -D_2'(p_2^*) & 0 \end{bmatrix}\right) \Big/ \det(HL)$$

$$= \left\{2D_2'(p_2^*) + \,(p_2^* - m - \lambda^*)\, D_2''(p_2^*)\right\} D_1'(p_2^*)^2 \big/ \det(HL) < 0$$

The picture looks like Fig. 1 above: As the tax rises from τ to τ', the Marginal revenue curve in the first country declines. The effect is to lower quantity sold in country 1, increase quantity sold in country 2, and reduce the Lagrangian multiplier. Since price is inversely related to quantity, we know that p_1 increases while p_2 decreases.

The above approach applies the implicit function theorem mechanically. Many professors approach the problem differently, by totally differentiating the Lagrangian FOC's. In fact, there is *absolutely no difference at all* between the two approaches: the total differentiation route simply involves a few extra steps. To demonstrate this, I'll solve the problem the long way, and show that we end up at the same place. The starting point for the total differentiation route is to recognize that the three endogenous variables are all functions of the exogenous variable τ. We can rewrite the FOC of the Lagrangian to emphasize this dependence.

$$
\begin{aligned}
L_{p_1(\tau)} &= D_1(p_1(\tau)) + (p_1(\tau) - \tau - m - \lambda(\tau))\, D_1'(p_1(\tau)) &= 0 \\
L_{p_2(\tau)} &= D_2(p_2(\tau)) + (p_2(\tau) \quad - m - \lambda(\tau))\, D_2'(p_2(\tau)) &= 0 \\
L_{\lambda(\tau)} &= Q - D_1(p_1(\tau)) - D_2(p_2(\tau)) &= 0
\end{aligned}
$$

We now totally differentiate each of the three equations above with respect to τ:

$$
\begin{aligned}
0 &= \left(2D_1'(p_1) + (p_1 - \tau - m - \lambda)\, D_1''(p_1)\right)\frac{dp_1}{d\tau} + 0\frac{dp_2}{d\tau} - D_1'(p_1)\frac{d\lambda}{d\tau} - D_1'(p_1)\frac{d\tau}{d\tau} \\
0 &= 0\frac{dp_1}{d\tau} + \left(2D_2'(p_2) + (p_2 - m - \lambda)\, D_2''(p_2)\right)\frac{dp_2}{d\tau} - D_2'(p_2)\frac{d\lambda}{d\tau} + 0\frac{d\tau}{d\tau} \\
0 &= -D_1'(p_1)\frac{dp_1}{d\tau} - D_2'(p_2)\frac{dp_1}{d\tau} + 0\frac{d\lambda}{d\tau} + 0\frac{d\tau}{d\tau}
\end{aligned}
$$

or, moving the $\frac{d\tau}{d\tau}$ terms to the left hand side and noting that $\frac{d\tau}{d\tau} \equiv 1$

$$
\begin{aligned}
D_1'(p_1) &= \left(2D_1'(p_1) + (p_1 - \tau - m - \lambda)\, D_1''(p_1)\right)\frac{dp_1}{d\tau} + 0\frac{dp_2}{d\tau} - D_1'(p_1)\frac{d\lambda}{d\tau} \\
0 &= 0\frac{dp_1}{d\tau} + \left(2D_2'(p_2) + (p_2 - m - \lambda)\, D_2''(p_2)\right)\frac{dp_2}{d\tau} - D_2'(p_2)\frac{d\lambda}{d\tau} \\
0 &= -D_1'(p_1)\frac{dp_1}{d\tau} - D_2'(p_2)\frac{dp_1}{d\tau} + 0\frac{d\lambda}{d\tau}
\end{aligned}
$$

or, in matrix notation

$$
\begin{bmatrix} D_1'(p_1) \\ 0 \\ 0 \end{bmatrix}
=
\begin{bmatrix}
2D_1'(p_1) + (p_1 - \tau - m - \lambda)\, D_1''(p_1) & 0 & -D_1'(p_1) \\
0 & 2D_2'(p_2) + (p_2 - m - \lambda)\, D_2''(p_2) & -D_2'(p_2) \\
-D_1'(p_1) & -D_2'(p_2) & 0
\end{bmatrix}
\begin{bmatrix} \frac{dp_1}{d\tau} \\ \frac{dp_2}{d\tau} \\ \frac{d\lambda}{d\tau} \end{bmatrix}
$$

or, multiplying both sides by the inverse of the 3×3 matrix

$$
\begin{bmatrix} \frac{dp_1}{d\tau} \\ \frac{dp_2}{d\tau} \\ \frac{d\lambda}{d\tau} \end{bmatrix}
=
\begin{bmatrix}
2D_1'(p_1) + (p_1 - \tau - m - \lambda)\, D_1''(p_1) & 0 & -D_1'(p_1) \\
0 & 2D_2'(p_2) + (p_2 - m - \lambda)\, D_2''(p_2) & -D_2'(p_2) \\
-D_1'(p_1) & -D_2'(p_2) & 0
\end{bmatrix}^{-1}
\begin{bmatrix} D_1'(p_1) \\ 0 \\ 0 \end{bmatrix}
$$

which is exactly the same expression as (1), which we obtained by applying the implicit function formula mechanically.

7.3.3. *Cournot question: ARE prelim, 2004, #4.*

Question:

a) Assume that there are $N > 2$ identical firms with constant marginal cost w, initially competing in a Cournot game in quantities. Suppose that total demand is $P = a - BQ$, where $Q = \sum_{i=1}^{n} q_i$. Derive the quantities produced, and show that the profits of each of the ith firm is given by $\pi_i = \frac{(a-w)^2}{B(N+1)^2}$.

b) Suppose that $M < N$ of these firms merge into a single firm that also has a constant marginal cost of w and chooses quantity as a Cournot competitor. What are the profits of these $N - M + 1$ firms?

c) Show that the non-merged firms are always better off but that the merged firms may in some cases be worse off.

d) Set $M = 2$ and $N = 3$. Suppose that upstream there is a single monopolist that supplies the downstream firms by choosing a wholesale cost w. We have two possible scenarios downstream:
   - with probability α the $K = N-M+1 = 2$ downstream firms operating after the merger remain competing as Cournot and will each choose the quantity $q_i(K) = \frac{a-w}{B(K+1)}$.
   - with probability $1-\alpha$ the game becomes a Stackelberg game, where the leader chooses $q_L(K) = \frac{a-w}{2B}$. and the follower chooses $q_F(K) = \frac{a-w}{4B}$.

   (1) Please setup the condition that would help you determine whether the upstream monopolist perfers there to be a merger and K firms downstream or no merger and have $N = 3$ downstream

   (2) How does the relative preference depend on α.

Answer:

a) Since firms are identical they will produce the same quantity in equilibrium. Hence the profit function for the i'th firm is $(a - (q_i + \sum_{j\neq i} q_j)B - w)q_i$. The first order condition for this problem is

$$0 = (a - (q_i + \sum_{j \neq i} q_j)B - w) - Bq_i$$

but since $q_i = q_j$, we have

$$\begin{aligned} 0 &= (a - Nq_iB - w) - Bq_i \\ &= (a - w) - (N+1)Bq_i \end{aligned}$$

so that the optimal quantity for firm i is

$$q_i^* = \frac{(a-w)}{(N+1)B}$$

Plugging this value back into the profit function we obtain

$$\begin{aligned} \pi_i &= \left((a-w) - N\frac{(a-w)}{(N+1)B}B\right)\frac{(a-w)}{(N+1)B} \\ &= \frac{(a-w)}{(N+1)}\frac{(a-w)}{(N+1)B} \\ &= \frac{(a-w)^2}{(N+1)^2B} \end{aligned}$$

b) When the M firms merge and act as a single Cournot firm, the *only* thing that changes is that the number of firms in the industry is reduced by $M-1$. (This is only true under the assumptioun of constant marginal cost.) All firms act still identically, so the previous analysis applies, once you change the number of firms. Each firm, including the merged one, earns profit of $\pi_i = \frac{(a-w)^2}{(N-M+2)^2B}$. Note, however, that for the merged firm, this profit now has to be split M ways.

c) For a non-merged firm, profit increases because the denominator in the profit function is reduced from $(N+1)$ to $N-M+2$. For the merged firm, there are two effects on profit. The per-firm profit is higher (price increases), but this profit has to be split M ways. Profit *for each* of the merged firms will increase iff the reduction by $M-1$ in the number of firms increases *per-firm* by a factor that exceeds $M-1$, that is, if $\sqrt{M}(N-M+2) < (N+1)$. For example, if $N = 101$ and $M = 100$, we have $\sqrt{M}(N-M+2) = 30 < (N+1) = 102$ (so each of the merged firms does better), but if $N = 10$ and $M = 4$, we have $\sqrt{M}(N - M + 2) = 16 > (N + 1) = 11$ (so each of the merged firms does worse),

d) (1) Let $EQ_M(\alpha)$ and Q_N denote total (expected) output under the merger and no-merger situations, respectively. Let w_M^* and w_N^* denote the corresponding wholesale prices. The upstream supplier will strictly prefer the merger iff $w_M^* EQ_M(\alpha) > w_N^* Q_N$. Clearly, $Q_N = \frac{3(a-w_N)}{4B}$, while

$$\begin{aligned} EQ_M(\alpha) &= \alpha\frac{2(a-w_M)}{3B} + (1-\alpha)\left(\frac{a-w_M}{2B} + \frac{a-w_M}{4B}\right) \\ &= \alpha\frac{2(a-w_M)}{3B} + (1-\alpha)\frac{3(a-w_M)}{4B} \end{aligned}$$

That is under the merger, firms produce strictly less then pre-merger if they act as Cournot, and the same in aggregate if they act as Stackelberg. Thus, the upstream producer strictly prefers the no-merger situation provided that $\alpha > 0$.

- The upstream monopolist's choice depends on α only in the sense that she's indifferent if $\alpha = 0$. But she weakly prefers the no-merger for all $\alpha \in [0, 1]$.

7.3.4. *Optimization question: ARE prelim, 2003, #2.*

Question: Suppose that aluminum has two uses. The first is for military aircraft, which has a constant elasticity of demand ϵ_1. The second is for lawn furniture, which has a constant elasticity of demand ϵ_2, where $\epsilon_1 < \epsilon_2$ in magnitured. Both products have fixed proportion functions $q_1 = \min[x/\beta_x, a/\beta_a]$, and $q_2 = \min[x/\gamma_x, a/\gamma_a]$, where a is the amount of aluminum ingot and x is the amount of all other goods. Aluminum has a constant marginal cost of production, m. Let $\bar{p}_i$ be the price under perfect competition of product i, for $i = 1, 2$. Normalize the price of other goods to unity. Let w be the price of aluminum ingot a.

a) Derive the cost functions of both products
b) Suppose that aluminum is producted by a monopolist who is also vertically integrated into aircraft and lawn funiture. Determine the monopolist's optimal prices p_1^* and p_2^* for both products.
c) Suppose an un-integrated aluminum monopolist could vertically integrate forward into one of the downstream products. Which product would it choose?
d) What is the monopolist's optimal price for aluminum ingots, w^*, and what is the optimal price of the product that it produces as an integrated firm (the integrated product).

Answer:

a) Obviously, other goods and aluminum must be in the proportion $x = a\beta_x/\beta_a$. One unit of almunimum ingot and β_x/β_a of the other goods produces $1/\beta_a$ units of aircraft. So q_1 units of aircraft require $q_1\beta_a$ of ingot and $q_1\beta_x$ of other inputs. Hence the cost function for aircraft is $C_1(q_1) = q_1(w\beta_a + \beta_x)$. Similarly, for lawn furniture the cost function is $C_2(q_2) = q_2(w\gamma_a + \gamma_x)$.
b) The demand for good $\#i$ is $Q_i = A_i p_i^{\epsilon_i}$, for $i = 1, 2$. The cost of aluminum to the integrated monopolist is m. So the profit function for the integrated monopolist is

$$\begin{aligned}\Pi(p_1, p_2) &= A_1 p_1^{\epsilon_1}(p_1 - (m\beta_a + \beta_x)) \quad + \quad A_2 p_2^{\epsilon_2}(p_2 - (m\gamma_a + \gamma_x)) \\ &= A_1 p_1^{1+\epsilon_1} - A_1 p_1^{\epsilon_1}(m\beta_a + \beta_x) \quad + \quad A_2 p_2^{1+\epsilon_2} - A_2 p_2^{\epsilon_2}(m\gamma_a + \gamma_x)\end{aligned}$$

The first order conditions are

$$\begin{aligned}\text{FOC}_1 &= A_1 p_1^{\epsilon_1}\left(\epsilon_1 + 1 - \epsilon_1(m\beta_a + \beta_x)/p_1\right) &= 0\\ \text{FOC}_2 &= A_2 p_2^{\epsilon_2}\left(\epsilon_2 + 1 - \epsilon_2(m\gamma_a + \gamma_x)/p_2\right) &= 0\end{aligned}$$

so that, providing $\epsilon < -1$ (check the SOC to see what happens if this condition is violated) the optimal prices are

$$\begin{aligned}p_1^* &= \frac{\epsilon_1(m\beta_a + \beta_x)}{\epsilon_1 + 1}\\ p_2^* &= \frac{\epsilon_2(m\gamma_a + \gamma_x)}{\epsilon_2 + 1}\end{aligned}$$

It's important for the remaining questions that because there is no resource constraint in this problem, the optimal prices are independent of each other, i.e., if you only got to control one of the prices and not the other, your choice of the price over which you had control would be the same.

c) Even though we don't explicitly solve for w^* until part d), we'll assume in this part that the price w of aluminum is set by the monopolist to be optimal for the market that the firm *doesn't* integrate into. We'll write w_i^* for the optimal value of w if the monopolist *doesn't* integrate into market i. I'll further assume that this market is perfectly competitive, so that the output price is determined by the condition that price equals marginal cost, i.e., if the aircraft market is competitive, then $\bar{p}_1 = w_1^*\beta_a + \beta_x$, while if the lawn furniture market is competitive, then $\bar{p}_2 = w_2^*\gamma_a + \gamma_x$. If the monopolist integrates forward into aircraft, her profit is

$$\Pi(p_1^*, w_2^*) = A_1(p_1^*)^{1+\epsilon_1} - A_1(p_1^*)^{\epsilon_1}(m\beta_a + \beta_x) \quad + \quad (w_2^* - m)\gamma_a A_2 \bar{p}_2^{\epsilon_2}$$

if she integrates forward into lawn furniture, her profit is

$$\Pi(p_2^*, w_1^*) = A_2(p_2^*)^{1+\epsilon_2} - A_2(p_2^*)^{\epsilon_2}(m\gamma_a + \gamma_x) \quad + \quad (w_1^* - m)\beta_a A_1 \bar{p}_1^{\epsilon_1}$$

She'll choose the first option if $\Pi(p_1^*, w_2^*) > \Pi(p_2^*, w_1^*)$ and the second if $\Pi(p_1^*, w_2^*) < \Pi(p_2^*, w_1^*)$.

d) The first important thing to note about this setup is that the choices of output price in one market and input price in the other are entirely independent of each other. There's no tradeoff between the two markets, because the monopolist doesn't have to bear in the market she monopolizes the price she sets in the market where she sells to third parties. Let's first assume she chooses to integrate into the aircraft industry. (It's extremely hard to believe that lawn furniture could be more profitable than aircraft. Whoever heard of a lawn-furniture-industrial complex?) In this case, her profit function is

$$\Pi(p_1, w) = A_1(p_1)^{1+\epsilon_1} - A_1(p_1)^{\epsilon_1}(m\beta_a + \beta_x) \quad + \quad (w-m)\gamma_a A_2(w\gamma_a + \gamma_x)^{\epsilon_2}$$

for which the first order conditions are

$$\mathrm{FOC}_{p_1} \quad = \quad A_1 p_1^{\epsilon_1}\left(\epsilon_1 + 1 - \epsilon_1(m\beta_a + \beta_x)/p_1\right) \quad = \quad 0$$

which is the same as in part b) and c), and, substituting for $\bar{p}_2$

$$\begin{aligned}
\mathrm{FOC}_w &= \gamma_a A_2(w\gamma_a + \gamma_x)^{\epsilon_2} \quad + \quad \gamma_a\epsilon_2(w-m)\gamma_a A_2(w\gamma_a+\gamma_x)^{\epsilon_2-1} \quad = \quad 0 \\
&= \frac{\gamma_a A_2}{(w\gamma_a+\gamma_x)^{1-\epsilon_2}}\Big((w\gamma_a+\gamma_x) \quad + \quad \gamma_a\epsilon_2(w-m)\Big) \\
&= \frac{\gamma_a A_2}{(w\gamma_a+\gamma_x)^{1-\epsilon_2}}\Big(w\gamma_a(1+\epsilon_2) \quad + \quad (\gamma_x - m\gamma_a\epsilon_2)\Big)
\end{aligned}$$

so that, presuming $m\gamma_a\epsilon_2 > \gamma_x$,

$$w_2^* \quad = \quad \frac{m\gamma_a\epsilon_2 - \gamma_x}{\gamma_a(1+\epsilon_2)}$$

By symmetry, if she chooses to integrate into the lawn furniture industry, the optimal price to charge for aluminum is

$$w_1^* \quad = \quad \frac{m\beta_a\epsilon_1 - \beta_x}{\beta_a(1+\epsilon_1)}$$

Plugging all these prices in the profit function, we obtain

$$\begin{aligned}
\Pi(p_1^*, w_2^*) &= A_1\left(\frac{\epsilon_1(m\beta_a+\beta_x)}{1+\epsilon_1}\right)^{1+\epsilon_1} - A_1\left(\frac{\epsilon_1(m\beta_a+\beta_x)}{1+\epsilon_1}\right)^{\epsilon_1}(m\beta_a+\beta_x) + (w_2^*-m)\gamma_a A_2\left(w_2^*\gamma_a+\gamma_x\right)^{\epsilon_2} \\
&= A_1\left(\frac{\epsilon_1(m\beta_a+\beta_x)}{1+\epsilon_1}\right)^{1+\epsilon_1}\left[1-\frac{1-\epsilon_1}{\epsilon_1}\right] + (w_2^*-m)\gamma_a A_2\left(w_2^*\gamma_a+\gamma_x\right)^{\epsilon_2} \\
&= A_1\left(\frac{\epsilon_1(m\beta_a+\beta_x)}{1+\epsilon_1}\right)^{1+\epsilon_1}\left[\frac{2\epsilon_1-1}{\epsilon_1}\right] + \left(\frac{m\gamma_a\epsilon_2-\gamma_x}{\gamma_a(1+\epsilon_2)}-m\right)\gamma_a A_2\left(\frac{m\gamma_a\epsilon_2-\gamma_x}{\gamma_a(1+\epsilon_2)}\gamma_a+\gamma_x\right)^{\epsilon_2} \\
&= A_1\left(\frac{\epsilon_1(m\beta_a+\beta_x)}{1+\epsilon_1}\right)^{1+\epsilon_1}\left[\frac{2\epsilon_1-1}{\epsilon_1}\right] + \left(\frac{m\gamma_a\epsilon_2-\gamma_x-m\gamma_a(1+\epsilon_2)}{\gamma_a(1+\epsilon_2)}\right)\gamma_a A_2\left(\frac{m\gamma_a\epsilon_2-\gamma_x+\gamma_x(1+\epsilon_2)}{(1+\epsilon_2)}\right) \\
&= A_1\left(\frac{\epsilon_1(m\beta_a+\beta_x)}{1+\epsilon_1}\right)^{1+\epsilon_1}\left[\frac{2\epsilon_1-1}{\epsilon_1}\right] - \left(\frac{\gamma_x+m\gamma_a}{\gamma_a(1+\epsilon_2)}\right)\gamma_a A_2\left(\frac{\epsilon_2(\gamma_x+m\gamma_a)}{(1+\epsilon_2)}\right)^{\epsilon_2} \\
&= A_1\left(\frac{\epsilon_1(m\beta_a+\beta_x)}{1+\epsilon_1}\right)^{1+\epsilon_1}\left[\frac{2\epsilon_1-1}{\epsilon_1}\right] - {\epsilon_2}^{\epsilon_2}A_2\left(\frac{\gamma_x+m\gamma_a}{1+\epsilon_2}\right)^{1+\epsilon_2}
\end{aligned}$$

similarly

$$\Pi(p_2^*, w_1^*) = A_2\left(\frac{\epsilon_2(m\gamma_a+\gamma_x)}{1+\epsilon_2}\right)^{1+\epsilon_2}\left[\frac{2\epsilon_2-1}{\epsilon_2}\right] - {\epsilon_1}^{\epsilon_1}A_1\left(\frac{\beta_x+m\beta_a}{1+\epsilon_1}\right)^{1+\epsilon_1}$$

From these two expressions, the relative profitability of the two markets depends in a complicated way on all of the parameters defining each of them.

7.4. **Manipulating first order conditions using the Implicit Function Theorem.**

Consider the standard utility maximization problem: maximize $u(\boldsymbol{z})$, subject to $\mathbf{p}\cdot\boldsymbol{z} = y$, where $\boldsymbol{z}, \mathbf{p} \in \mathbb{R}^2$, Set up the Lagrangian $L(\bar{\mathbf{p}}, \bar{y}; \boldsymbol{z}, \lambda) = u(\boldsymbol{z}) + \lambda(y - \mathbf{p}\cdot\boldsymbol{z})$. I.e., initially, $\mathbf{p}$ and y are parameters.

- Solve for the first order conditions of the Lagrangian. That is, solve for $(\bar{\boldsymbol{z}}, \bar{\lambda})$ such that $\bigtriangledown L(\bar{\mathbf{p}}, \bar{y}; \bar{\boldsymbol{z}}, \bar{\lambda}) = 0$. (Here the gradient consists of the partial derivatives w.r.t. the $\boldsymbol{z}$'s and λ, not w.r.t. to the paramers $\mathbf{p}$ and y. Note that since the constraint must hold with equality, we must have $\frac{\partial L}{\partial \lambda} = 0$.)
- observe that what we've done is find one point on a particular level set of $\bigtriangledown L$.

- Now consider variations in the exogenous variables $\mathbf{p}$ and y and think about the *level set* $\{(\boldsymbol{z}, \lambda, \mathbf{p}, y) : \bigtriangledown L(\cdot,\cdot,\cdot,\cdot) = 0\}$.
- Specifically, observe that $\bigtriangledown L : \mathbb{R}^6 \to \mathbb{R}^3$, so that we can solve for first three variables in terms of the last three.
- To convert this problem to the format of the general implicit function theorem, what's f, $\mathbf{x}$, α, g?
  - $f = \bigtriangledown L$.
  - look at the level set $\bigtriangledown L = 0$.
  - $\mathbf{x} = (z_1, z_2, \lambda)$
  - $\boldsymbol{\alpha} = (p_1, p_2, y)$
  - $\boldsymbol{g}(\boldsymbol{\alpha}) \equiv \boldsymbol{g}(p_1, p_2, y) \equiv ((z_1(p_1, p_2, y), z_2(p_1, p_2, y), \lambda(p_1, p_2, y))$

Now mindlessly apply the implicit function theorem to compute $\frac{\partial z_i(\cdot)}{\partial p_j}$, etc. That is:

$$\begin{bmatrix} \frac{\partial g^1(\cdot)}{\partial p_1} & \frac{\partial g^1(\cdot)}{\partial p_2} & \frac{\partial g^1(\cdot)}{\partial y} \\ \frac{\partial g^2(\cdot)}{\partial p_1} & \frac{\partial g^2(\cdot)}{\partial p_2} & \frac{\partial g^2(\cdot)}{\partial y} \\ \frac{\partial g^3(\cdot)}{\partial p_1} & \frac{\partial g^3(\cdot)}{\partial p_2} & \frac{\partial g^3(\cdot)}{\partial y} \end{bmatrix} = -\,\mathrm{J}(\bigtriangledown \mathrm{L})^{-1}_{(\boldsymbol{z},\lambda)} \begin{bmatrix} \frac{\partial L_{z_1}(\cdot)}{\partial p_1} & \frac{\partial L_{z_1}(\cdot)}{\partial p_2} & \frac{\partial L_{z_1}(\cdot)}{\partial y} \\ \frac{\partial L_{z_2}(\cdot)}{\partial p_1} & \frac{\partial L_{z_2}(\cdot)}{\partial p_2} & \frac{\partial L_{z_2}(\cdot)}{\partial y} \\ \frac{\partial L_{\lambda}(\cdot)}{\partial p_1} & \frac{\partial L_{\lambda}(\cdot)}{\partial p_2} & \frac{\partial L_{\lambda}(\cdot)}{\partial y} \end{bmatrix}$$

where

$$\mathrm{J}(\bigtriangledown \mathrm{L})_{(\boldsymbol{z},\lambda)} = \begin{bmatrix} L_{z_1 z_1}(\cdot) & L_{z_1 z_2}(\cdot) & L_{z_1 \lambda}(\cdot) \\ L_{z_2 z_1}(\cdot) & L_{z_2 z_2}(\cdot) & L_{z_2 \lambda}(\cdot) \\ L_{\lambda z_1}(\cdot) & L_{\lambda z_2}(\cdot) & L_{\lambda\lambda}(\cdot) \end{bmatrix}.$$

For example, consider computing the Slutsky equation by differentiating the first order conditions of the Lagrangian. Recall that the lagrangian is: $L(\bar{\mathbf{p}}, \bar{y}; \boldsymbol{z}, \lambda) = u(\boldsymbol{z}) + \lambda(y - \mathbf{p}\cdot\boldsymbol{z})$. So that the first order conditions are:

$$\bigtriangledown L = \begin{bmatrix} L_{z_1} \\ L_{z_2} \\ L_{\lambda} \end{bmatrix} = \begin{bmatrix} u_1 - \lambda p_1 \\ u_2 - \lambda p_2 \\ y - p_1 z_1 - p_2 z_2 \end{bmatrix} = 0.$$

Hence

$$\text{HL} = \text{J}(\triangledown \text{L})_{(\boldsymbol{z},\lambda)} = \begin{bmatrix} u_{11} & u_{12} & -p_1 \\ u_{21} & u_{22} & -p_2 \\ -p_1 & -p_2 & 0 \end{bmatrix}.$$

and we have

$$\begin{bmatrix} \frac{\partial z_1}{\partial p_1} & \frac{\partial z_1}{\partial p_2} & \frac{\partial z_1}{\partial y} \\ \frac{\partial z_2}{\partial p_1} & \frac{\partial z_2}{\partial p_2} & \frac{\partial z_2}{\partial y} \\ \frac{\partial \lambda}{\partial p_1} & \frac{\partial \lambda}{\partial p_2} & \frac{\partial \lambda}{\partial y} \end{bmatrix} = -\,\text{J}(\triangledown \text{L})^{-1}_{(\boldsymbol{z},\lambda)} \begin{bmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ -z_1 & -z_2 & 1 \end{bmatrix}$$

$$= \text{J}(\triangledown \text{L})^{-1}_{(\boldsymbol{z},\lambda)} \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ z_1 & z_2 & -1 \end{bmatrix}.$$

If you like, rewrite this expression as three equations, each of which is of the form $A\mathbf{x} = \mathbf{b}$, e.g.,

$$\begin{bmatrix} u_{11} & u_{12} & -p_1 \\ u_{21} & u_{22} & -p_2 \\ -p_1 & -p_2 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial z_1}{\partial p_1} \\ \frac{\partial z_2}{\partial p_1} \\ \frac{\partial \lambda}{\partial p_1} \end{bmatrix} = \begin{bmatrix} \lambda \\ 0 \\ z_1 \end{bmatrix}.$$

and now apply Cramer's Rule, e.g.,

$$\frac{\partial z_1}{\partial p_1} = \det(\text{J}(\triangledown \text{L})_{(\boldsymbol{z},\lambda)})^{-1} \det\left(\begin{bmatrix} -\lambda & u_{12} & -p_1 \\ 0 & u_{22} & -p_2 \\ -z_1 & -p_2 & -0 \end{bmatrix} \right).$$

Here's a second example,which is conceptually identical to the preceding one. Consider a competitive firm, producing a single output y from a vector of inputs $\mathbf{x} \in \mathbb{R}^n$ using a technology f. Let p denote the output price and $\mathbf{w}$ denote the vector of input prices. Given input and output prices, the first order condition for profit maximization is

$$\mathsf{FOC}(p, \mathbf{w}, \mathbf{x}) = p \triangledown f(\mathbf{x}) - \mathbf{w} = 0 \tag{2}$$

where (2) is of course an n-vector of equalities. To guarantee a local maximum we require that $Hf(\mathbf{x})$ is negative definite. Usefully, this implies that the determinant of $Hf(\mathbf{x})$ is nonzero, and this turns out to be the condition we need in order to be able to use the implicit function theorem.

Our task is to derive the *input demand functions*, i.e., $\mathbf{x}(\mathbf{w})$. To find this we mindlessly apply the theorem to the level set (2) to obtain $Jx(\mathbf{w})$, i.e.,

$$\begin{aligned} Jx(\mathbf{w}) &= -\left(J\mathsf{FOC}_{\mathbf{x}}(p, \mathbf{w}, \mathbf{x})\right)^{-1} \times J\mathsf{FOC}_{\mathbf{w}}(p, \mathbf{w}, \mathbf{x}) \\ &= -\left(pHf(p, \mathbf{w}, \mathbf{x})\right)^{-1} \times (-I^n) \\ &= \left(pHf(p, \mathbf{w}, \mathbf{x})\right)^{-1} \times I^n \end{aligned}$$

where where I^n is the n-dimensional identity function, $J\mathsf{FOC}_{\mathbf{x}}$ denotes the Jacobian of FOC treating $(p, \mathbf{w})$ as parameters, and $J\mathsf{FOC}_{\mathbf{w}}$ denotes the Jacobian of FOC treating $(p, \mathbf{x})$ as parameters.