Contents

4. Foundations of Comparative Statics (cont)

Key points of this lecture:

(1) The solution to *any* economic system can be characterized as the level set corresponding to zero of *some* function.
(2) When you do comparative statics analysis of a problem, you are studying the slope of the level set that characterizes the problem.
(3) The implicit function theorem tells you
  (a) when this slope is well defined
  (b) if it is well-defined, what are the derivatives of the implicit function

(4) Implicit function theorem (single variable version): Given $f : \mathbb{R}^2 \to \mathbb{R}^1$ continuously differentiable and $(\bar{\alpha}, \bar{x}) \in \mathbb{R}^2$, if $\frac{\partial f(\bar{\alpha},\bar{x})}{\partial x} \neq 0$, then there exist neighborhoods U^α of $\bar{\alpha}$ and U^x of $\bar{x}$ and a continuously differentiable function $g : U^\alpha \to U^x$ such that for all $\alpha \in U^\alpha$,

$$\begin{aligned} f(\alpha, g(\alpha)) &= f(\bar{\alpha}, \bar{x}) \text{ i.e., } (\alpha, g(\alpha)) \text{ is on the level set of } f \text{ through } (\bar{\alpha}, \bar{x}) \\ g'(\alpha) &= -\frac{\frac{\partial f(\alpha,g(\alpha))}{\partial \alpha}}{\frac{\partial f(\alpha,g(\alpha))}{\partial x}} \end{aligned}$$

(5) Implicit function theorem (intermediate version): Given $f : \mathbb{R}^{n+1} \to \mathbb{R}^1$ continuously differentiable and $(\bar{\boldsymbol{\alpha}}, \bar{x}) \in \mathbb{R}^n \times \mathbb{R}^1$ if $f_{n+1}(\bar{\boldsymbol{\alpha}}, \bar{x}) \neq 0$, then there exist neighborhoods $U^{\boldsymbol{\alpha}}$ of $\bar{\boldsymbol{\alpha}}$ and U^x of $\bar{x}$ and a continuously differentiable function $g : U^{\boldsymbol{\alpha}} \to U^x$ such that for all $\boldsymbol{\alpha} \in U^{\boldsymbol{\alpha}}$,

$$\begin{aligned} f(\boldsymbol{\alpha}, g(\boldsymbol{\alpha})) &= f(\bar{\boldsymbol{\alpha}}, \bar{x}) \text{ i.e., } g \text{ puts us on the level set of } f \text{ containing } (\bar{\boldsymbol{\alpha}}, \bar{x}) \\ g_j(\boldsymbol{\alpha}) &= -f_j(\boldsymbol{\alpha}, g(\boldsymbol{\alpha}))/f_{n+1}(\boldsymbol{\alpha}, g(\boldsymbol{\alpha})). \end{aligned}$$

(6) Implicit function theorem (final version): Theorem: Given $\mathbf{f} : \mathbb{R}^{n+m} \to \mathbb{R}^m$ continuously differentiable, and $(\bar{\boldsymbol{\alpha}}, \bar{\mathbf{x}}) \in \mathbb{R}^n \times \mathbb{R}^m$, if the determinant of $\mathrm{Jf}_{\mathbf{x}}(\bar{\boldsymbol{\alpha}}, \bar{\mathbf{x}}))$ is not zero, then there exist neighborhoods $U^{\boldsymbol{\alpha}}$ of $\bar{\boldsymbol{\alpha}}$ and $U^{\mathbf{x}}$ of $\bar{\mathbf{x}}$ and a continuously differentiable function $\mathbf{g} : U^{\boldsymbol{\alpha}} \to U^{\mathbf{x}}$ such that for all $\boldsymbol{\alpha} \in U^{\boldsymbol{\alpha}}$,

$$\mathbf{f}(\boldsymbol{\alpha}, \mathbf{g}(\boldsymbol{\alpha})) = \mathbf{f}(\bar{\boldsymbol{\alpha}}, \bar{\mathbf{x}}) \quad \text{i.e., } \mathbf{g} \text{ puts us on the level set of } \mathbf{f} \text{ containing } (\bar{\boldsymbol{\alpha}}, \bar{x}) \text{ and}$$

$$\begin{bmatrix} \frac{\partial g^1(\boldsymbol{\alpha})}{\partial \alpha_1} & \frac{\partial g^1(\boldsymbol{\alpha})}{\partial \alpha_2} & \cdots & \frac{\partial g^1(\boldsymbol{\alpha})}{\partial \alpha_n} \\ \frac{\partial g^2(\boldsymbol{\alpha})}{\partial \alpha_1} & \frac{\partial g^2(\boldsymbol{\alpha})}{\partial \alpha_2} & \cdots & \frac{\partial g^2(\boldsymbol{\alpha})}{\partial \alpha_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g^m(\boldsymbol{\alpha})}{\partial \alpha_1} & \frac{\partial g^m(\boldsymbol{\alpha})}{\partial \alpha_2} & \cdots & \frac{\partial g^m(\boldsymbol{\alpha})}{\partial \alpha_n} \end{bmatrix} = -\,\mathrm{Jf}_{\mathbf{x}}(\boldsymbol{\alpha}, \mathbf{g}(\boldsymbol{\alpha}))^{-1} \begin{bmatrix} \frac{\partial f^1(\boldsymbol{\alpha},\mathbf{g}(\boldsymbol{\alpha}))}{\partial \alpha_1} & \frac{\partial f^1(\boldsymbol{\alpha},\mathbf{g}(\boldsymbol{\alpha}))}{\partial \alpha_2} & \cdots & \frac{\partial f^1(\boldsymbol{\alpha},\mathbf{g}(\boldsymbol{\alpha}))}{\partial \alpha_n} \\ \frac{\partial f^2(\boldsymbol{\alpha},\mathbf{g}(\boldsymbol{\alpha}))}{\partial \alpha_1} & \frac{\partial f^2(\boldsymbol{\alpha},\mathbf{g}(\boldsymbol{\alpha}))}{\partial \alpha_2} & \cdots & \frac{\partial f^2(\boldsymbol{\alpha},\mathbf{g}(\boldsymbol{\alpha}))}{\partial \alpha_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f^m(\boldsymbol{\alpha},\mathbf{g}(\boldsymbol{\alpha}))}{\partial \alpha_1} & \frac{\partial f^m(\boldsymbol{\alpha},\mathbf{g}(\boldsymbol{\alpha}))}{\partial \alpha_2} & \cdots & \frac{\partial f^m(\boldsymbol{\alpha},\mathbf{g}(\boldsymbol{\alpha}))}{\partial \alpha_n} \end{bmatrix}.$$

4.3. **Implicit function Theorem.**

Level sets and comparative statics: The solution to *any* economic system can be characterized as the level set of some function. The members of the level set are pairs, consisting of endogenous and exogenous variables; each such pair produces the same value for our function of interest. By convention, once we've identified the function whose level sets we are interested in, we focus on its *zero* level set.

Here's a simple economic model: $S = S(t,p)$, $D = D(y,p)$, $S = D$, where p denotes market price, t denotes a tax rate paid by the producer and y denotes consumer income level. The solution to this model can be represented as the level set $f(t,y,p) \equiv 0$, where $f = S - D$, $\boldsymbol{\alpha} = t, y$ and $x = p$. Given a pair of exogenous variables (t,y), we are interested in the p value such that the triple (p,t,y) belongs to the level set of f corresponding to zero.

When we do comparative statics on this problem, we start out at some initial solution to the problem, (t^*, y^*, p^*), then change either t, y, or both, and ask: how does p have to change in order to stay on this zero level set. In other words, what's the *slope* of the relevant level set at the starting point (t^*, y^*, p^*).

Cf, elementary micro-economics: you have an isoquant, you start out at a point (k^*, ℓ^*), then you change k and ask "how much does ℓ have to change to keep you on the same level set?"; the answer is the marginal rate of technical substitution, which is the *slope* of the isoquant at the point (k^*, ℓ^*). Every comparative statics exercise you'll ever do is *exactly* analogous to this elementary exercise: you have a starting point, you change an exogenous variable or variables and adjust some endogenous variable or variables so that you stay on the given level set.

In the first example we looked at, the defining property of the economic system was market equilibrium. Another class of economic system arises from optimization. In this case, the level set that our exog-endog pairs live on is the level set corresponding to zero of the *first order conditions of the Lagrangian.* E.g., consider the problem $\max_{\mathbf{x}} u(\mathbf{x})$ s.t. $\mathbf{p} \cdot \mathbf{x} \leq y$. The Lagrangian for this problem is $L(\mathbf{x}, \lambda; \mathbf{p}, y) = u(\mathbf{x}) + \lambda(y - \mathbf{p} \cdot \mathbf{x})$. Assuming the income constraint is binding, the first order conditions necessary conditions for a solution are $f = \bigtriangledown_{(\mathbf{x},\lambda)} L = 0$. Once again, when we do comparative statics on this problem, we change one of the three exogenous variables (prices and/or income), and ask: "how do the three endogenous variables have to change to keep us on the same level set of f?". The answer gives you the slopes of the components of the consumer's demand function.

The mathematical tool that's used to compute slopes of the level set is the *implicit function theorem.* To motivate the content of the theorem, go back to the isoquant of a production function. You have $q = f(k,\ell)$. We're interested in the slope $\frac{d\ell}{dk}|_{f(k,\ell)=\bar{q}}$ at some point (k^*, ℓ^*) such that $f(k^*, \ell^*) = \bar{q}$. Specifically, we'll consider the Cobb Douglas production function $f = k^{\alpha_k} \ell^{\alpha_\ell}$. The brute force way to compute $\frac{d\ell}{dk}|_{f(k,\ell)=\bar{q}}$ would be to manipulate the equation $\bar{q} = k^{\alpha_k} \ell^{\alpha_\ell}$, until ℓ is on the left hand side; Then we'd have ℓ as an *explicit* function of k and $\bar{q}$. We'd then obtain $\frac{d\ell}{dk}|_{f(k,\ell)=\bar{q}}$ as the partial derivative of $\ell(\bar{q}, k)$ w.r.t. k.

But of course we never ever go to the trouble of computing $\ell(\cdot,\cdot)$ explicitly. Rather, we've memorized that $\frac{d\ell}{dk} = -\frac{f_k}{f_\ell}$. That is, *the slope of the explicit function we're interested in is a combination of the partial derivatives of the function whose level set we are required to stay on.* That is, instead of computing explicitly the function $\ell(\bar{q}, k)$, we just write down its slope in terms of the primitive function whose level set we are interested in. The implicit function theorem makes all this precise. It's called the implicit function theorem because the function whose slope we're calculating is *implicit* rather than *explicit* in our analysis.

The implicit function theorem has two parts: first, under what conditions can you do what I've just described; second, if you can do it, what's the formula relating the slope you are interested in to the partial derivatives of the function whose level set you are given.

Implicit function theorem: motivation. Given a level set of a function $f : \mathbb{R}^2 \to \mathbb{R}$ *and a starting point in* $\mathbb{R}^2$, the implicit function theorem tell us

(1) when we can characterize the level set *locally* (i.e., in a nbd of the starting point) as the *graph* of a *differentiable* function from α to x? That is, is there an alternative way to represent $\{(\alpha, x) : f(\alpha, x) = c\}$ as the graph of some function $x = g(\alpha)$?
(2) what the slope is of this function, i.e., $\frac{dx}{d\alpha}$.

Look at the next diagram:

(1) In Case 1, it's obvious that x is a perfectly nice, well behaved function of α.
(2) In Case 2, x is certainly a function of α, but it is not so well-behaved. Specifically, it isn't differentiable, because $f_x(\bar{\alpha}, \bar{x}) = 0$, so that the "slope" of the implicit function is infinite at this point.
(3) Case 3 is a little more subtle: we can't write x globally as a function of α. What we can do, however, is to restrict our attention to a small neighborhood of (α, x), and characterize the level set *restricted to that neighborhood* as the graph of a function mapping α to x. Note also that we can't *always* do this: there is *no* neighborhood of the point $(\bar{\alpha}, \bar{x})$, on which the level set can be represented as the graph of a function.
(4) The next point is about computation. In some instances, we could solve for x explicitly, then take derivative of this function that we've computed. For example, suppose $u(\alpha, x) = \sqrt{\alpha x}$ and we are interested in the the level set $\sqrt{\alpha x} = 4.5$. We could solve this to get $x = g(\alpha)$, then take the derivative. The implicit function theorem saves us the trouble: we can can calculate $g'(.)$ directly from the derivatives of f, without ever constructing the function explicitly.

Implicit function theorem (single variable version): Given $f : \mathbb{R}^2 \to \mathbb{R}^1$ continuously differentiable and $(\bar{\alpha}, \bar{x}) \in \mathbb{R}^2$, if $\frac{\partial f(\bar{\alpha}, \bar{x})}{\partial x} \neq 0$, then there exist neighborhoods U^α of $\bar{\alpha}$ and U^x of $\bar{x}$ and a continuously differentiable function $g : U^\alpha \to U^x$ such that for all $\alpha \in U^\alpha$,

$$\begin{aligned} f(\alpha, g(\alpha)) &= f(\bar{\alpha}, \bar{x}) \text{ i.e., } (\alpha, g(\alpha)) \text{ is on the level set of } f \text{ through } (\bar{\alpha}, \bar{x}) \\ g'(\alpha) &= -\frac{\frac{\partial f(\alpha, g(\alpha))}{\partial \alpha}}{\frac{\partial f(\alpha, g(\alpha))}{\partial x}} \end{aligned}$$

Note that there are no bar's on the α's on the second line. The reason is that g is an implicitly defined *function*, and its derivative is defined *everywhere* on the nbd U^α by the above condition. To reiteratre, $g'(\alpha)$ is the slope of the *function* that *locally* represents the level set through $(\bar{\alpha}, \bar{x})$.

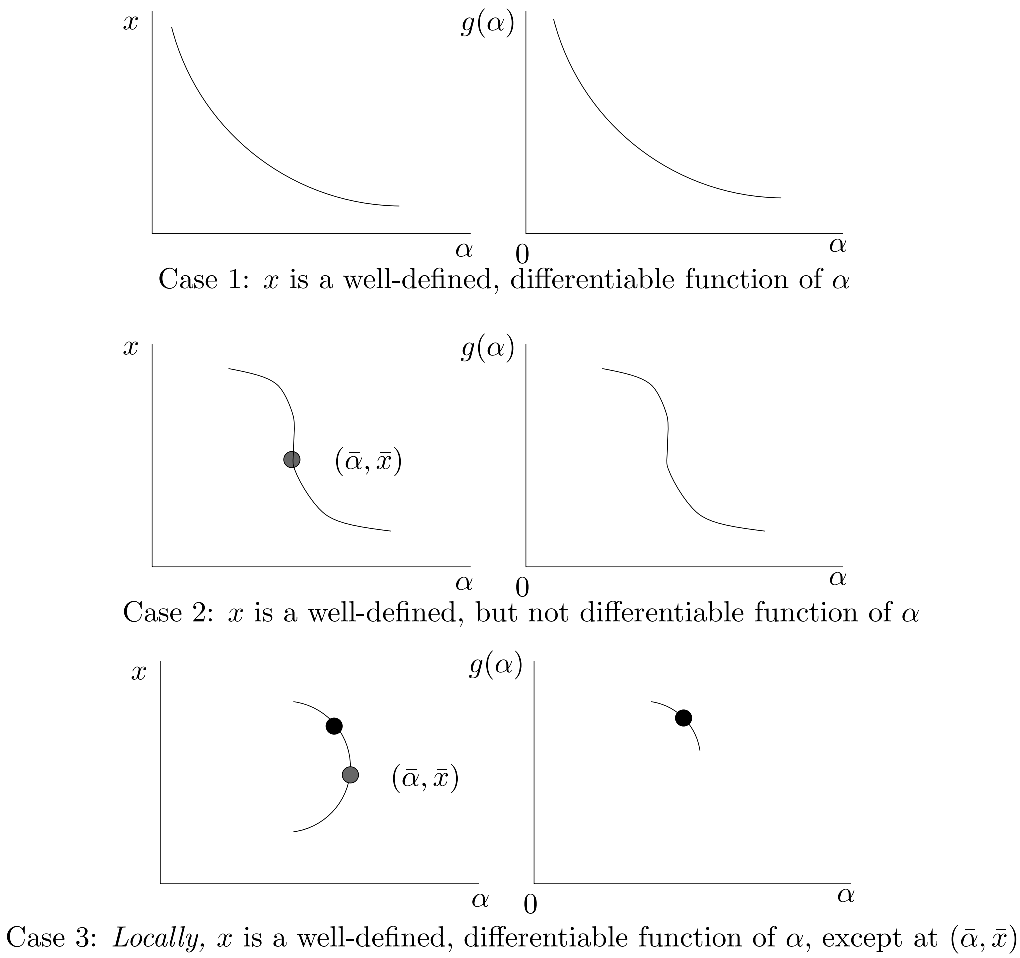

FIGURE 1. Deriving x as a function of α from level sets

The proof of this result is completely trivial:

$$\frac{df(\alpha, g(\alpha))}{d\alpha} \equiv 0 = \frac{\partial f(\alpha, g(\alpha))}{\partial \alpha} + \frac{\partial f(\alpha, g(\alpha))}{\partial x} g'(\alpha)$$

Hence

$$g'(\alpha) = -\frac{\frac{\partial f(\alpha, g(\alpha))}{\partial \alpha}}{\frac{\partial f(\alpha, g(\alpha))}{\partial x}}$$

But in fact there are lots of subtleties that you don't think about if you just prove the theorem mechanically. They do, however, become apparent from the picture.

- The neighborhood condition: in general, the condition that (α, x) lies on a given level set doesn't necessarily *globally* associate a unique x to α.
  - Look at the bottom left panel of Fig. 1. Except where the level set is vertical, there are two x's associated with each α value: thus, for each α value, the implicit function theorem identifies *two* functions, quite different from each other, each of which represents a different piece of the same level set.
  - Once you know both x and α, however, then you can almost always identify a little neighborhood on which there is a unique relationship between x and α.
- The condition on $\frac{\partial f(\bar{\alpha},\bar{x})}{\partial x}$:
  - Given the formula, it's obvious that you can't have $\frac{\partial f(\bar{\alpha},\bar{x})}{\partial x} = 0$, else you wouldn't be able to define the ratio.
  - What does the nonzero derivative caveat mean in the statement of theorem mean? If the level set is vertical at $(\bar{\alpha}, \bar{x})$, then *there isn't even a neighborhood* on which the fact that $(\bar{\alpha}, \bar{x})$ lies on a given level set implies a unique relationship between x and α. Verticality of the level set is vertical at $\bar{x}$ is precisely the condition that $f_x(\bar{\alpha}, \bar{x}) = 0$. For example, let $f(\alpha, x) = \alpha^2 + x^2$; in this case, the level sets are circles; in particular, the level set corresponding to $f = 1$ is the unit circle; observe that $f_2(\alpha, x) = 2x = 0$, when $x = 0$.
  - Hence the appropriate condition to ensure that x is *locally* uniquely determined given α is $f_2(\bar{\alpha}, \bar{x}) \neq 0$.

Here's an example using more familiar notation: the computation of the MRS. We have a utility function $u : \mathbb{R}^2 \to \mathbb{R}^1$, and want to know the slope of an indifference curve through $(\bar{z}_1, \bar{z}_2)$. In this case, fitting our specific example into the general notation of the implicit function theorem,

- α is z_1,
- x is z_2.
- $f(\alpha, x)$ is $u(z_1, z_2) - u(\bar{z}_1, \bar{z}_2)$.

hence we have $f(\bar{z}_1, \bar{z}_2) = 0$, and we want to vary α (in our example, z_1) and see how x (in our example, z_2) has to change in order to keep us on the level set of f corresponding to 0.

We'll write z_2 as a function of z_1, i.e., at this point,

$$\left.\frac{dz_2}{dz_1}\right|_{u \text{ constant}} = -\frac{\frac{\partial u(\bar{z}_1,\bar{z}_2)}{\partial z_1}}{\frac{\partial u(\bar{z}_1,\bar{z}_2)}{\partial z_2}}$$

Here's another example that illustrates the computational value of the theorem. $f(\alpha, x) = \alpha x^{15} + \alpha^{13} + x^{95}$;

- it would be clearly extremely difficult to write down g explicitly.
- however, it's trivial to compute the slope of g.
- $f_1(\alpha, x) = x^{15} + 13\alpha^{12}$
  $f_2(\alpha, x) = 15\alpha x^{14} + 95x^{94}$;

- Implicit function theorem says that $g'(\alpha) = -f_1(\alpha, g(.))/f_2(\alpha, g(.)) = -\frac{(x^{15}+13\alpha^{12})}{(\alpha x^{14}+95x^{94})}$

Implicit function Theorem: intermediate version: As with the other important concepts in the course, the implicit function theorem can be stated in various degrees of generality. We now go up one level and assume that f has $n+1$ arguments.

Theorem: Given $f : \mathbb{R}^{n+1} \to \mathbb{R}^1$ continuously differentiable and $(\bar{\boldsymbol{\alpha}}, \bar{x}) \in \mathbb{R}^n \times \mathbb{R}^1$ if $f_{n+1}(\bar{\boldsymbol{\alpha}}, \bar{x}) \neq 0$, then there exist neighborhoods $U^{\boldsymbol{\alpha}}$ of $\bar{\boldsymbol{\alpha}}$ and U^x of $\bar{x}$ and a continuously differentiable function $g : U^{\boldsymbol{\alpha}} \to U^x$ such that for all $\boldsymbol{\alpha} \in U^{\boldsymbol{\alpha}}$,

$$\begin{aligned} f(\boldsymbol{\alpha}, g(\boldsymbol{\alpha})) &= f(\bar{\boldsymbol{\alpha}}, \bar{x}) \text{ i.e., } g \text{ puts us on the level set of } f \text{ containing } (\bar{\boldsymbol{\alpha}}, \bar{x}) \\ g_j(\boldsymbol{\alpha}) &= -f_j(\boldsymbol{\alpha}, g(\boldsymbol{\alpha}))/f_{n+1}(\boldsymbol{\alpha}, g(\boldsymbol{\alpha})). \end{aligned}$$

In words, implicit function theorem says that if you have one equation in $n+1$ unknowns, you can solve for *any one* of the unknowns in terms of the other n, provided that...

Proof again is a trivial exercise in differentiation: since

$$f(\boldsymbol{\alpha}, g(\boldsymbol{\alpha})) \equiv f(\bar{\boldsymbol{\alpha}}, \bar{x})$$

we can take the partial derivative of f w.r.t. α_j:

$$\frac{df(\boldsymbol{\alpha}, g(\boldsymbol{\alpha}))}{d\alpha_j} = 0 = \frac{\partial f(\boldsymbol{\alpha}, g(\boldsymbol{\alpha}))}{\partial \alpha_j} + \frac{\partial f(\boldsymbol{\alpha}, g(\boldsymbol{\alpha}))}{\partial x} \frac{\partial g(\boldsymbol{\alpha}))}{\partial \alpha_j}$$

rearranging yields:

$$\frac{\partial g(\boldsymbol{\alpha}))}{\partial \alpha_j} = -\frac{\frac{\partial f(\boldsymbol{\alpha}, g(\boldsymbol{\alpha}))}{\partial \alpha_j}}{\frac{\partial f(\boldsymbol{\alpha}, g(\boldsymbol{\alpha}))}{\partial x}}$$

An important feature to note is that the domain of f has one more dimension than the domain of g. Reason is that it is the *graph* of g, i.e., $\{(\boldsymbol{\alpha}, g(\boldsymbol{\alpha})) : \boldsymbol{\alpha} \in U^{\boldsymbol{\alpha}}\}$ that locally recovers the level set. That is, the graph of a real valued function is a set that lives in a Euclidean space one dimension higher than the dimension of the domain of the function. In this case, a point $(\boldsymbol{\alpha}, g(\boldsymbol{\alpha}))$ is a point on the level set of f.

Recall that I began the lecture saying that the solution to *any* economic system can be represented as the level set of some function. Recall the simple economic model that I wrote down: $S = S(t, p)$, $D = D(y, p)$, $S = D$, where p denotes market price, t denotes a tax rate paid by the producer and y denotes consumer income level. The solution to this model can be represented as the level set $f(\boldsymbol{\alpha}, x) \equiv 0$, where $f = S - D$, $\boldsymbol{\alpha} = t, y$ and $x = p$. The level set of f corresponding to zero is the set of all (price , tax , income) triples such that the price clears the market for the corresponding values of the exogenous variables.

Explicitly we have the following relationship

$$\boldsymbol{\alpha} = (t, y) \tag{1}$$

$$x = p \tag{2}$$

$$f(\boldsymbol{\alpha}, x) = S(t,p) - D(y,p) \tag{3}$$
$$g(\boldsymbol{\alpha}) = p(t,y) \tag{4}$$

$p(t,y)$ tells us how p must change with params to keep us on the level set $S(t,p) - D(y,p) = 0$.

Solve for an initial equilibrium $(\bar{\boldsymbol{\alpha}}, \bar{\mathbf{x}}) = (\bar{t}, \bar{y}, \bar{p})$ and compute

$$\frac{\partial p(\bar{t},\bar{y})}{\partial t} = -\frac{\frac{\partial f(\bar{t},\bar{y},\bar{p})}{\partial t}}{\frac{\partial f(\bar{t},\bar{y},\bar{p})}{\partial p}} = -\frac{\frac{\partial S(\bar{t},\bar{p})}{\partial t}}{\frac{\partial S(\bar{t},\bar{p})}{\partial p} - \frac{\partial D(\bar{y},\bar{p})}{\partial p}}$$

$$\frac{\partial p(\bar{t},\bar{y})}{\partial y} = -\frac{\frac{\partial f(\bar{t},\bar{y},\bar{p})}{\partial y}}{\frac{\partial f(\bar{t},\bar{y},\bar{p})}{\partial p}} = -\frac{\frac{\partial S(\bar{t},\bar{p})}{\partial y}}{\frac{\partial S(\bar{t},\bar{p})}{\partial p} - \frac{\partial D(\bar{y},\bar{p})}{\partial p}}$$

We can now estimate the effect of a shift in the parameter vector $(\bar{t}, \bar{y})$ on the equilibrium value of p, i.e., suppose we add a small vector $(\mathrm{dt}, \mathrm{dy})$ to the $(\bar{t}, \bar{y})$ and want to get an estimate of dp, the resultant change in p. To get an approximate answer, we evaluate, as usual, the differential at the magnitude of the change, and obtain

$$\mathrm{dp} = p_t(\bar{t},\bar{y})dt + p_y(\bar{t},\bar{y})dy \tag{5}$$
$$= -\left(\frac{\frac{\partial f(\bar{t},\bar{y},\bar{p})}{\partial t}}{\frac{\partial f(\bar{t},\bar{y},\bar{p})}{\partial p}}dt + \frac{\frac{\partial f(\bar{t},\bar{y},\bar{p})}{\partial y}}{\frac{\partial f(\bar{t},\bar{y},\bar{p})}{\partial p}}dy\right) \tag{6}$$

Note that since (6) is a *linear* function, we can estimate the constants $p_t(\bar{t}, \bar{y})$ and $p_y(\bar{t}, \bar{y})$ using a linear regression, i.e., one of the form

$$y = \beta' X + \epsilon$$

Implicit function Theorem: the most general version: The most general version says that if you have m equations in $n + m$ unknowns, you can solve for *any* m of the unknowns in terms of the other n, provided that the usual conditions are satisfied.

Interpretation: write your economic model in the form $f(\boldsymbol{\alpha}; \mathbf{x}) = 0$; solve for changes in $\mathbf{x} \in \mathbb{R}^m$ as a function of changes in the *parameter vector* $\boldsymbol{\alpha} \in \mathbb{R}^n$. In other words, think of $\boldsymbol{\alpha}$ as a vector of *exogenous variables* for your model, and of $\mathbf{x}$ as a vector of *endogenous variables*. We are typically interested in how the endogenous vector $\mathbf{x}$ changes as the exogenous vector $\boldsymbol{\alpha}$ changes. Write $f(\boldsymbol{\alpha}, \mathbf{g}(\boldsymbol{\alpha}))$ and use implicit function theorem to find the slope of $\mathbf{g}$ w.r.t. the components of $\boldsymbol{\alpha}$.

Note that the relative size of n and m is completely immaterial. So long as you have m equations and m endogenous variables, you can have as many or as few *exogenous* variables as you want.

Theorem: Given $\mathbf{f} : \mathbb{R}^{n+m} \to \mathbb{R}^m$ continuously differentiable, let

$$\mathrm{Jf}_{\mathbf{x}}(\boldsymbol{\alpha}, \mathbf{x}) = \begin{bmatrix} \frac{\partial f^1(\boldsymbol{\alpha};\mathbf{x})}{\partial x_1} & \frac{\partial f^1(\boldsymbol{\alpha};\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f^1(\boldsymbol{\alpha};\mathbf{x})}{\partial x_m} \\ \frac{\partial f^2(\boldsymbol{\alpha};\mathbf{x})}{\partial x_1} & \frac{\partial f^2(\boldsymbol{\alpha};\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f^2(\boldsymbol{\alpha};\mathbf{x})}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f^m(\boldsymbol{\alpha};\mathbf{x})}{\partial x_1} & \frac{\partial f^m(\boldsymbol{\alpha};\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f^m(\boldsymbol{\alpha};\mathbf{x})}{\partial x_m} \end{bmatrix}$$

$$= \begin{bmatrix} f^1_{n+1}(\boldsymbol{\alpha};\mathbf{x}) & f^1_{n+2}(\boldsymbol{\alpha};\mathbf{x}) & \cdots & f^1_{n+m}(\boldsymbol{\alpha};\mathbf{x}) \\ f^2_{n+1}(\boldsymbol{\alpha};\mathbf{x}) & f^2_{n+2}(\boldsymbol{\alpha};\mathbf{x}) & \cdots & f^2_{n+m}(\boldsymbol{\alpha};\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f^m_{n+1}(\boldsymbol{\alpha};\mathbf{x}) & f^m_{n+2}(\boldsymbol{\alpha};\mathbf{x}) & \cdots & f^m_{n+m}(\boldsymbol{\alpha};\mathbf{x}) \end{bmatrix}$$

where f^1_{n+1} denotes the $n+1$'th partial derivative of the function f^1, which is, in turn, the first of the m *single-valued* functions stacked on top of each other that make up the vector valued function $\mathbf{f}$. Given $(\bar{\boldsymbol{\alpha}}, \bar{\mathbf{x}}) \in \mathbb{R}^n \times \mathbb{R}^m$, if the determinant of $\mathrm{Jf}_{\mathbf{x}}(\bar{\boldsymbol{\alpha}}, \bar{\mathbf{x}}))$ is not zero, then there exist neighborhoods $U^{\boldsymbol{\alpha}}$ of $\bar{\boldsymbol{\alpha}}$ and $U^{\mathbf{x}}$ of $\bar{\mathbf{x}}$ and a continuously differentiable function $\mathbf{g} : U^{\boldsymbol{\alpha}} \to U^{\mathbf{x}}$ such that for all $\boldsymbol{\alpha} \in U^{\boldsymbol{\alpha}}$,

$$\mathbf{f}(\boldsymbol{\alpha}, \mathbf{g}(\boldsymbol{\alpha})) = \mathbf{f}(\bar{\boldsymbol{\alpha}}, \bar{\mathbf{x}}) \quad \text{i.e., } \mathbf{g} \text{ puts us on the level set of } f \text{ containing } (\bar{\boldsymbol{\alpha}}, \bar{x}), \text{ and}$$

$$\begin{bmatrix} \frac{\partial g^1(\boldsymbol{\alpha})}{\partial \alpha_1} & \frac{\partial g^1(\boldsymbol{\alpha})}{\partial \alpha_2} & \cdots & \frac{\partial g^1(\boldsymbol{\alpha})}{\partial \alpha_n} \\ \frac{\partial g^2(\boldsymbol{\alpha})}{\partial \alpha_1} & \frac{\partial g^2(\boldsymbol{\alpha})}{\partial \alpha_2} & \cdots & \frac{\partial g^2(\boldsymbol{\alpha})}{\partial \alpha_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g^m(\boldsymbol{\alpha})}{\partial \alpha_1} & \frac{\partial g^m(\boldsymbol{\alpha})}{\partial \alpha_2} & \cdots & \frac{\partial g^m(\boldsymbol{\alpha})}{\partial \alpha_n} \end{bmatrix} = -\,\mathrm{Jf}_{\mathbf{x}}(\boldsymbol{\alpha}, \mathbf{g}(\boldsymbol{\alpha}))^{-1} \begin{bmatrix} \frac{\partial f^1(\boldsymbol{\alpha},\mathbf{g}(\boldsymbol{\alpha}))}{\partial \alpha_1} & \frac{\partial f^1(\boldsymbol{\alpha},\mathbf{g}(\boldsymbol{\alpha}))}{\partial \alpha_2} & \cdots & \frac{\partial f^1(\boldsymbol{\alpha},\mathbf{g}(\boldsymbol{\alpha}))}{\partial \alpha_n} \\ \frac{\partial f^2(\boldsymbol{\alpha},\mathbf{g}(\boldsymbol{\alpha}))}{\partial \alpha_1} & \frac{\partial f^2(\boldsymbol{\alpha},\mathbf{g}(\boldsymbol{\alpha}))}{\partial \alpha_2} & \cdots & \frac{\partial f^2(\boldsymbol{\alpha},\mathbf{g}(\boldsymbol{\alpha}))}{\partial \alpha_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f^m(\boldsymbol{\alpha},\mathbf{g}(\boldsymbol{\alpha}))}{\partial \alpha_1} & \frac{\partial f^m(\boldsymbol{\alpha},\mathbf{g}(\boldsymbol{\alpha}))}{\partial \alpha_2} & \cdots & \frac{\partial f^m(\boldsymbol{\alpha},\mathbf{g}(\boldsymbol{\alpha}))}{\partial \alpha_n} \end{bmatrix}.$$

That is, we have an equality between an $m \times n$ matrix and an $m \times m$ matrix times an $m \times n$ matrix. Observe that when $m = 1$, this just collapses to the old implicit function theorem.

Fig. 2 illustrates what's going on. We saw four of the panels of this figure in the lecture math-Calculus3. Now they have an entirely different interpretation. Consider what the implicit function theorem tells us for a function $\mathbf{f} : \mathbb{R}^{n+m} \to \mathbb{R}^m$, where in this case, $m = 2$, and $n = 1$. The top two panels on the left represent, respectively, the level sets of f^1 and f^2 passing through $(\bar{\boldsymbol{\alpha}}, \bar{\mathbf{x}})$. The bottom two panels on the left translate the two tangent planes to the origin as usual. The top right panel plots the two tangent planes together. The bottom right panel is a zoom of the top right panel: its only purpose is to make clearer that the intersection of the two tangent planes starts out *below* the horizontal axis, then rises *above* it.

We know that as α changes, x_1 and x_2 have to change also, to keep us on the *two* level sets we've drawn. More precisely, since the implicit function theorem is (like everything else we do) all about first order approximations, they need to change in such a way as to keep us on the two *tangent planes.* How can this happen? This is where the two right panels of Fig. 2 are helpful. Clearly, x_1 and x_2 must change by dx_1 and dx_2, where these two changes lie in the intersection of the two tangent planes. Notice significantly that in order to remain "in the crack" between (i.e., in the intersection of) the two tangent planes, there's only one direction along the crack you can move. In our picture, for example, when α increases (decreases), *both* components of $\mathbf{x}$ have to increase (decrease) in order that we remain in the crack. In other words, for $i = 1, 2$, $\frac{dx_i}{d\alpha} > 0$.

How do we interpret the condition that $\det \mathrm{Jf}_{\mathbf{x}} \neq 0$. It takes pretty good 3-D eyesight to visualize it, but if you look hard, you'll see that $\det Jf_{\mathbf{x}} = 0$ *iff the intersection of the two tangent planes lives in the horizontal plane.* In this case, you could move in *either* direction along the crack from the origin and remain in the crack. This is the analog of the simple case we looked at when α and x were scalars: when $f_x = 0$, you could either increase or decrease x as α moved, and stay in the tangent line. Here, if $\det \mathrm{Jf}_{\mathbf{x}} = 0$, then both $\mathbf{dx}$ and $-\mathbf{dx}$ keep you in the crack.

As an example, consider a slightly more complex economic system, where $S^i = S^i(t, p_1 ... p_m)$, demand $D^i = D^i(y, p_1 ... p_m)$, etc:

$$\begin{aligned}\boldsymbol{\alpha} &= (t, y)\\ \mathbf{x} &= \mathbf{p}\\ f^i(\boldsymbol{\alpha}, \mathbf{x}) &= S^i(t, \mathbf{p}) - D^i(y, \mathbf{p})\\ \mathbf{g}(\boldsymbol{\alpha}) &= \mathbf{p}(t, y)\end{aligned}$$

$\mathbf{p}(t, y)$ tells us how $\mathbf{p}$ must change to keep us on the level set $\mathbf{S}(t, \mathbf{p}) - \mathbf{D}(y, \mathbf{p}) = \mathbf{0}$.

Solve for an initial equilibrium $(\bar{\boldsymbol{\alpha}}, \bar{\mathbf{x}}) = (\bar{t}, \bar{y}, \bar{\mathbf{p}})$ define

$$\mathrm{Jf}_{\mathbf{p}}(\bar{t}, \bar{y}, \bar{\mathbf{p}}) = \begin{bmatrix} \frac{\partial f^1(\bar{t},\bar{y},\bar{\mathbf{p}})}{\partial p_1} & \frac{\partial f^1(\bar{t},\bar{y},\bar{\mathbf{p}})}{\partial p_2} & \cdots & \frac{\partial f^1(\bar{t},\bar{y},\bar{\mathbf{p}})}{\partial p_m} \\ \frac{\partial f^2(\bar{t},\bar{y},\bar{\mathbf{p}})}{\partial p_1} & \frac{\partial f^2(\bar{t},\bar{y},\bar{\mathbf{p}})}{\partial p_2} & \cdots & \frac{\partial f^2(\bar{t},\bar{y},\bar{\mathbf{p}})}{\partial p_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f^m(\bar{t},\bar{y},\bar{\mathbf{p}})}{\partial p_1} & \frac{\partial f^m(\bar{t},\bar{y},\bar{\mathbf{p}})}{\partial p_2} & \cdots & \frac{\partial f^m(\bar{t},\bar{y},\bar{\mathbf{p}})}{\partial p_m} \end{bmatrix}$$

and compute

$$\begin{bmatrix} \frac{\partial g^1(\boldsymbol{\alpha})}{\partial \alpha_1} \\ \frac{\partial g^2(\boldsymbol{\alpha})}{\partial \alpha_1} \\ \vdots \\ \frac{\partial g^m(\boldsymbol{\alpha})}{\partial \alpha_1} \end{bmatrix} = \begin{bmatrix} \frac{\partial p^1(\bar{t},\bar{y})}{\partial t} \\ \frac{\partial p^2(\bar{t},\bar{y})}{\partial t} \\ \vdots \\ \frac{\partial p^m(\bar{t},\bar{y})}{\partial t} \end{bmatrix} = -\,\mathrm{Jf}_{\mathbf{p}}(\bar{t}, \bar{y}, \bar{\mathbf{p}})^{-1} \begin{bmatrix} \frac{\partial f^1(\bar{t},\bar{y},\bar{\mathbf{p}})}{\partial t} \\ \frac{\partial f^2(\bar{t},\bar{y},\bar{\mathbf{p}})}{\partial t} \\ \vdots \\ \frac{\partial f^m(\bar{t},\bar{y},\bar{\mathbf{p}})}{\partial t} \end{bmatrix} \qquad \text{etc...}$$

4.4. **A last point.**

Often one has to take the derivatives of the derivatives you get from the implicit function theorem. This always caused me a lot of anxiety; do I need to use the implicit function theorem twice? Answer is, once you have the derivatives from the IFT, then they are *just like* any other derivatives. If you need the derivatives of $\bigtriangledown g$, where g is defined implicitly, you just do what you always do.

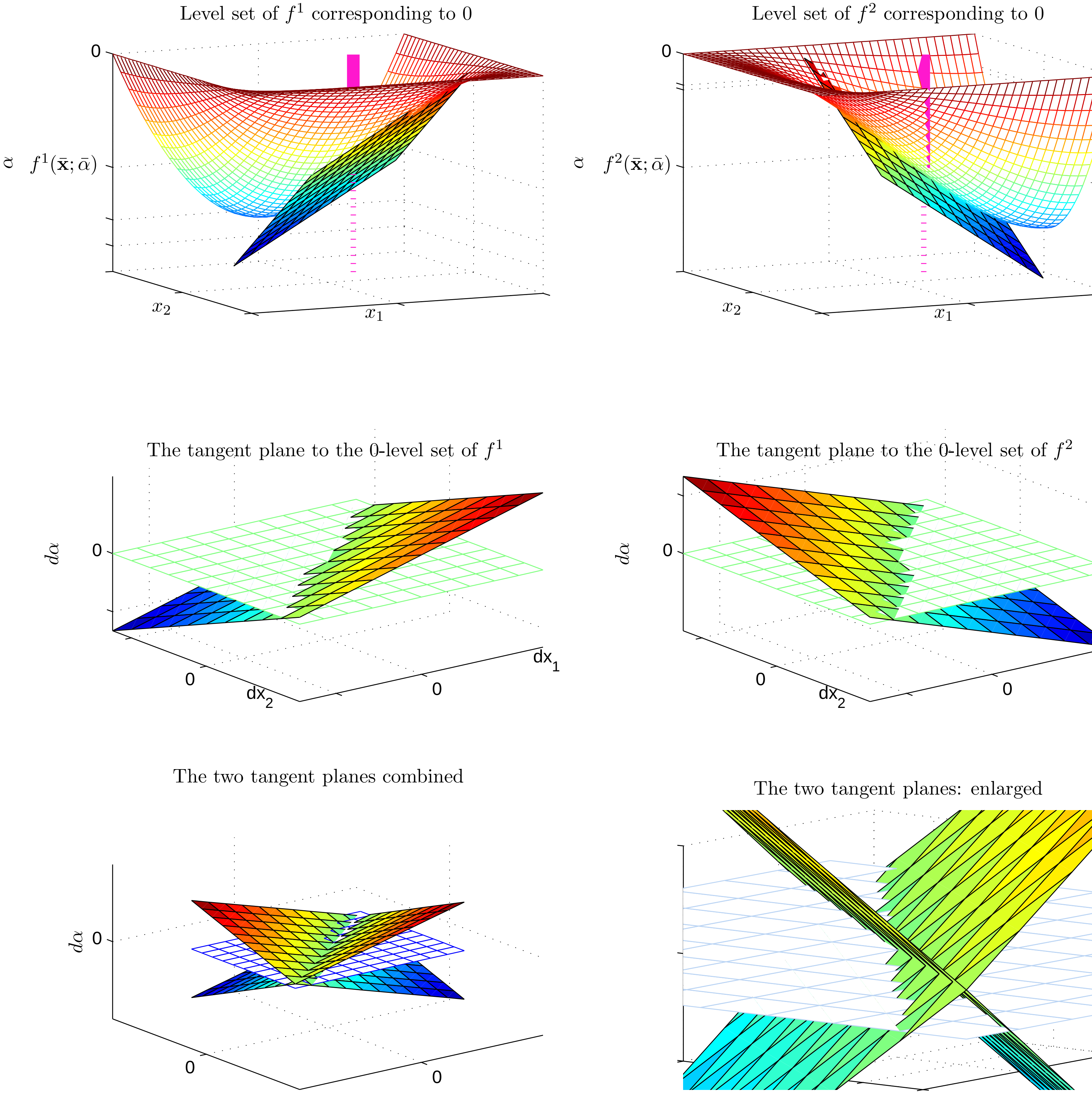

FIGURE 2. The implicit function theorem: how $\mathbf{x}$ moves when α moves