ARE211, Fall2015

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Contents

4. Foundations of Comparative Statics

Overview of the Topic

(1) **Implicit function theorem**: used to compute relationship between endogenous and exogenous variables.
  (a) Context: First order conditions of an optimization problem. Examples
    - e.g., demands as a function of income
    - e.g., input demands as a function of output
  (b) Context: Some kind of system equilibrium. Examples:
    - market equilibrium prices as a function of economy endowments
    - Cournot quantities as a function of demand parameter

(2) **Envelope theorem**: simplifies computation of change in value function (optimized value of objective function) as parameters change.

(a) Context: First order conditions of an optimization problem (always)

(b) Examples:

(i) optimized utility as a function of income

(ii) maximized profit as a function of price

(3) **Envelope theorem** (bonus): in very special cases you can short-circuit the implicit function theorem, and use the envelope theorem to get the same information. that the implicit function theorem would give you, but much more easily

(a) Context: first order conditions of an optimization problem (always)

(b) Examples: input demand functions (Hotelling's Lemma)

(i) input demand functions (Hotelling's Lemma)

(ii) conditional input demand functions (Shephards's Lemma)

Key points of this lecture:

(1) The envelope theorem for unconstrained maximization:

if $\mathbf{x}^*(\mathbf{b})$ solves $\max_{\mathbf{x}} f(\mathbf{b},\mathbf{x})$, then $\frac{df(\mathbf{b},\mathbf{x}^*(\mathbf{b}))}{db_k} = \frac{\partial f(\mathbf{b},\mathbf{x}^*(\mathbf{b}))}{\partial b_k}$. i.e., the *total* derivative of f w.r.t. b_k equals the *partial* derivative of f w.r.t. b_k.

(2) The envelope theorem for constrained maximization:

if $(\mathbf{x}^*(\mathbf{b}),\boldsymbol{\lambda}^*(\mathbf{b}))$ solves $\max_{\mathbf{x}} f(\mathbf{b},\mathbf{x})$ s.t. $h^j(\mathbf{b},\mathbf{x}) = 0$, $j = 1,...m$, then

$$\frac{df(\mathbf{b},\mathbf{x}^*(\mathbf{b}))}{db_k} \quad = \quad \frac{\partial f(\mathbf{b},\mathbf{x}^*(\mathbf{b}))}{\partial b_k} \quad + \quad \sum_{j=1}^{m} \lambda_j^*(\mathbf{b})\frac{\partial h^j(\mathbf{b},\mathbf{x}^*(\mathbf{b}))}{\partial b_k}$$

i.e., the *total* derivative of f w.r.t. b_k equals the *partial* derivative of f w.r.t. b_k plus the $\boldsymbol{\lambda}^*$-weighted sum of the *partial* derivatives of the h^j's w.r.t. b_k.

(3) Example of the unconstrained envelope theorem (Hotelling's lemma):

Let $\pi^*(p,\mathbf{w}) = pf(\mathbf{x}^*) - \mathbf{w}\cdot\mathbf{x}^*$ be the maximized value of profits given output price p and input price vector $\mathbf{w}$. Then the i'th input demand function is $x_i^*(\cdot) = -\frac{\partial \pi^*(\cdot,\cdot)}{\partial w_i}$.

(4) Example of the constrained envelope theorem (Shephard's lemma):

Let $\hat{c}(\bar{q},p,\mathbf{w}) = \mathbf{w}\cdot\hat{\mathbf{x}}$ be the minimized level of costs given prices $(p,\mathbf{w})$ and output level $\bar{q}$. Then the i'th conditional input demand function is $\hat{x}_i(\cdot) = \frac{\partial \hat{c}(\cdot,\cdot,\cdot)}{\partial w_i}$.

(5) Another example of the constrained envelope theorem (LRATC and SRATC):

The long-run average total cost curve (with all inputs variable) is the *outer envelope* of the short-run average total cost curves (with some inputs fixed); when fixed inputs are at their optimal levels, the slopes of LRATC and SRATC curves are equal.

4.1. **The envelope theorem for unconstrained maximization**

In economics, we're often interested in a function which has two arguments; the second is a function of the first. As we've discussed in a previous lecture (CALCULUS2), economists typically deal with this by invoking the (unfortunately named) concept of the total derivative: if $f(b, x) = f(b, x(b))$, then *total derivative* of f w.r.t b is $df/db = f_b(b, x(b)) + f_x(b, x(b))x'(b)$, where b and x are here scalars. When b changes in this case, there is a change in f due to two factors: first b changes, also, x changes as b changes.

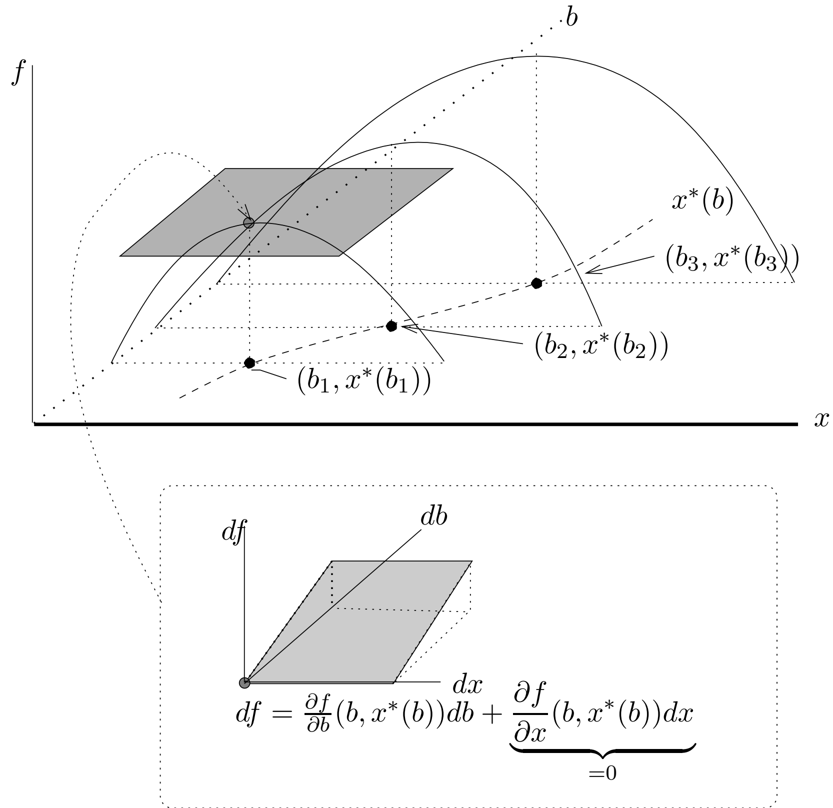
In this lecture, we'll consider the case in which the second argument is a *special kind* of function of b; $x^*(\cdot)$ is the value of x that maximises (or minimizes) f for each value of b. The function $f(b, x^*(b))$ is then called the *value function.*

Example $\pi(p, q^*(p))$; for each p, pick the q that maximizes profits for that p; call this function $q^*(p)$. Now ask how profit adjust as price changes and the producer adjusts quantity.

Other examples of value functions in economics are the expenditure function and the indirect utility function.

Answer is given by the envelope theorem which says that in this case, $df/db = \partial f/\partial b$. (i.e., you've learnt to tell the difference between df and ∂f; now you find that in this case, there isn't any difference.)

The envelope theorem: Varian: Given $f : \mathbb{R}^2 \to \mathbb{R}^1$ (differentiable) and a function $x^* : \mathbb{R}^1 \to \mathbb{R}^1$ (differentiable) defined by the condition that for each b, $x^*(b)$ maximizes $f(b, \cdot)$. Then the total derivative of the function $f(b, x^*(b))$ with respect to b is $df(\cdot, x^*(\cdot))/db = \partial f(\cdot, x^*(\cdot))/\partial b$.



When you evaluate df, the dx term has no effect because it is multiplied by zero

Figure 1. The envelope theorem and the differential

Mathematical proof is trivial

$$\frac{df(b, x^*(b))}{db} = f_b(b, x^*(b)) + f_x(b, x^*(b))\frac{dx^*(b)}{db}$$

Necessary condition for x to maximize $f(b, \cdot)$ is that $f_x(b, x) = 0$; this is how $x^*(b)$ is defined; hence $f_x(b, x^*(b)) = 0$ by definition of $x^*(b)$.

The picture is much more important; note that in the picture, the golden rule is broken, for display purposes: the first component of the function is pictured on the horizontal axis.

- The line in the domain $x^*(b)$ has the property that vertically above points on this line, the function $f(b,\cdot)$ is maximized in the x direction.
- Now as you move out along the line $x^*(b)$ there are in general two contributors to the change in f; f changes because b changes AND because x changes.
- In this particular case, f doesn't change when x changes, but it does change when b changes.
- *It's worth noting that there is nothing special about moving out along the line $x^*(b)$: if you start out at $(b, x^*(b))$ and move in any direction whatsoever, the only thing that matters is the movement in the b direction.*
- Compare this picture to the general case in which we move out at the same angle, where the change in f due to x is very substantial;
- In the general case, the x effect really makes a difference.

4.2. **The envelope theorem for constrained maximization**

The point of the above version of the envelope theorem is that when you are maximizing unconstrainedly a function $f(b,\cdot)$, where b is a parameter, then the rate of change in f as b changes does not depend on the rate at which $\mathbf{x}^*(\cdot)$ moves with b. (In class, I did the theorem for the case in which $\mathbf{x}^*$ was a scalar; it is also obviously true when $\mathbf{x}^*$ is a vector.)

An analogous result holds when you solve the problem:

$$\text{maximize } f(b,\mathbf{x}) \text{ subject to the constraint that } h(b,\mathbf{x}) = 0. \tag{1}$$

Let $\mathbf{x}^*(b)$ denote the solution to (1) and let $M(b)$ denote the maximized value of f given b.

(Notice that there is a difference between (1) and the familiar specification, i.e., max $f(b, \mathbf{x})$ subject to $g(\mathbf{x}) = b$. But the familiar specification is a special case of the current one. To see this, let $h(b, \mathbf{x}) = b - g(\mathbf{x})$. The current specification allows for more general comparative statics than we have seen before. In our original specification i.e., max $f(b, \mathbf{x})$ subject to $g(\mathbf{x}) = b$, we learnt how to do comparative statics w.r.t. b, but not with respect to the other parameters of $g(\cdot)$. When we write the problem in the current form, we can do comparative statics w.r.t. *any* parameter of either the objective or the constraint. For example, suppose that our problem is $\max u(\mathbf{x})$ s.t. $\mathbf{p} \cdot \mathbf{x} = y$. Our analysis in the preceding section taught us how to do comparative statics w.r.t. y *but not w.r.t. the components of* $\mathbf{p}$. We are about to see how to do comparative statics w.r.t. these components as well.)

The constrained version of the envelope theorem says that $\frac{dM(b)}{db} = \frac{\partial f(b,\mathbf{x}^*(b))}{\partial b} + \lambda^*(b)\frac{\partial h(b,\mathbf{x}^*(b))}{\partial b}$. (Notice that for the special case in which f does not depend on b, so that $\frac{\partial f(b,\mathbf{x}^*(b))}{\partial b} = 0$, and $h(b, \mathbf{x}^*) = b - g(\mathbf{x}^*)$, so that $\frac{\partial h(b,\mathbf{x}^*(b))}{\partial b} = 1$, the expression $\frac{dM(b)}{db} = \frac{\partial f(b,\mathbf{x}^*(b))}{\partial b} + \lambda^*(b)\frac{\partial h(b,\mathbf{x}^*(b))}{\partial b}$ reduces to simply $\frac{dM(b)}{db} = \lambda^*(b)$, which is the old familiar result: $\lambda^*(\cdot)$ measures the rate at which the objective function increases as the constraint is relaxed.) As in the unconstrained version of the envelope theorem, the *total* derivative of M w.r.t. b does involve the partial derivatives of f w.r.t. the elements of the $\mathbf{x}^*$ vector, but these terms disappear in the expression for $\frac{dM(b)}{db}$.

The striking difference between the unconstrained and the constrained theorems is that in the former case, the movement in the x direction didn't matter because the $\frac{\partial f(b,\mathbf{x}^*(b))}{\partial x_i}$'s were zero. In the present case, the gradient of f *isn't* zero, and yet the movement in the x direction *still* doesn't matter. So while the unconstrained theorem is very easy to explain intuitively, the constrained theorem is by no means so.

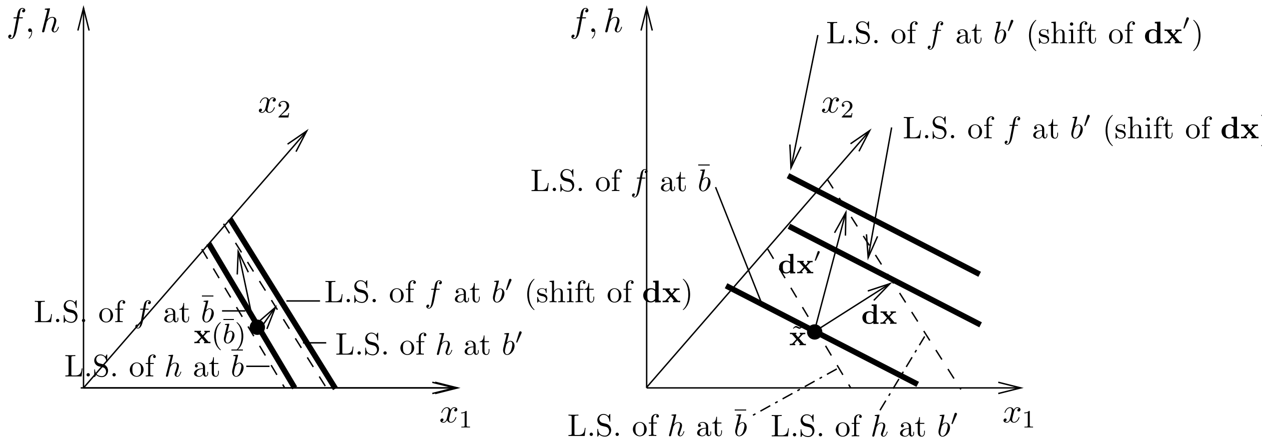


FIGURE 2. The envelope theorem: constrained optimization when Hessian is zero

Fig. 2 illustrates the theorem, for the simplest case in which there is only one constraint, $h(b, \mathbf{x}^*) = b - g(\mathbf{x}^*)$. To present the result in its sharpest form, the figure depicts a *linear* optimization problem, i.e., the level sets of both the objective f and the constraint h are affine functions. In the upper panel, we start out at an optimum $\mathbf{x}^*(\bar{b})$ on the constraint set, i.e., the level set of h associated with $\bar{b}$. (Note that for this linear case, the optimum is not unique). In the right panel, the level sets of the objective function have a different slope from that of the constraint function, so we start out at an arbitrary (non-optimal) point $\tilde{\mathbf{x}}^*$ on the constraint set. Now consider what happens when b changes to b', and the constraint line moves outwards. It is critical to my story that the new constraint line is parallel to the old one: this must be the case because g is affine, so its gradient cannot change direction (or length) with x. In the upper panel, the change in the objective function *doesn't depend on how* $\mathbf{x}^*$ *moves*, provided that $\mathbf{x}^*$ moves to the new, parallel constraint line. In the right panel, different directions of movement from the old constraint line to the new result in different changes in the objective function. This is a graphical depiction of the mathematical result that when you start out at a constrained optimum, and shift to the new, parallel constraint line, the change in the value of the objective depends on the partials of both f and h w.r.t. b *but not* on the partials of either f or h w.r.t. the components of $\mathbf{x}^*$.

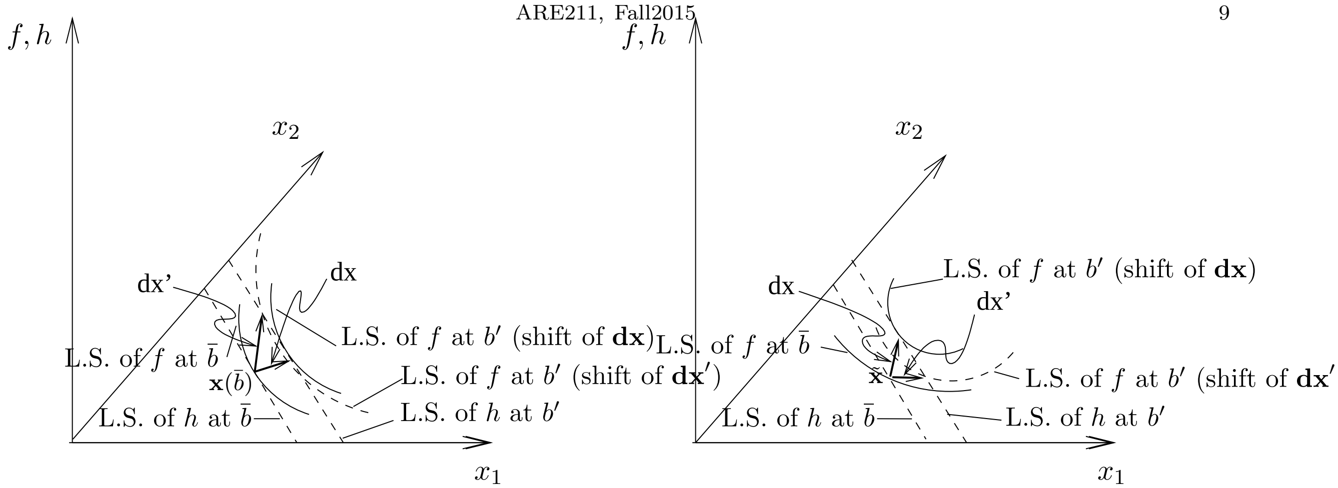


FIGURE 3. The envelope theorem: constrained optimization when Hessian is nonzero

When either the objective or the constraint are nonlinear, i.e., when you add some curvature to the level sets, it's a little less easy to see from pictures what's going on. Indeed the important implication of the envelope theorem seems to be false, since differences in the direction of movement $\mathbf{dx}$ *do* make a difference. However, as the two panels of Fig. 3 illustrate, the difference is much smaller when you start out at an optimum $\mathbf{x}^*(\bar{b})$ (as in the left panel of the figure) than when you start out at some arbitrary point $\tilde{\mathbf{x}^*}$ (as in the right panel of the figure). In both panels, you get to a higher level set when you move to the new constraint line in the direction $\mathbf{dx}$ of the gradient than if you move in a different direction, such as $\mathbf{dx}'$. In the left panel, however, the difference between the level sets you reach is only *second order*, i.e., due to the Hessian term, but in the right panel the difference is first order (i.e., due to the first order term in the Taylor expansion). As we've noted, Fig. 3 seems to contradict the envelope theorem, which says that the only things that matter are (a) how far out the constraint moves when b changes; (b) how rapidly f increases as you move out to the new constraint. What *doesn't* matter, according to the envelope theorem, is the *direction* in which you move in order to get to the new constraint line. The reconciliation of this paradox (at least for the case when only one constraint is binding—when two or more are binding we have to tell a more complicated story) is that the envelope theorem is only telling you about the *first* order Taylor approximation to the change in f when b changes. In Fig. 2, there *are no* second order effects, i.e., the Hessian is zero. So the first term in the Taylor expansion tells the whole story: movements in the $\mathbf{x}^*$ direction don't matter at all when you start out at an optimum. In the left panel of Fig. 3, there is no first order effect because we start out at an optimum, only second order effects. Since the envelope theorem just gives you a *first-order* approximation to the true change in f when b changes, the differences we see in the left panel of Fig. 3 evaporate. In the right panel of Fig. 3, we don't start from an optimum and there are both first and second order effects.

Now for the formalism. We'll do the general NPP, i.e., with m inequality constraints. Once again, we'll see that the direction in which the solution vector $\mathbf{x}^*(\cdot)$ moves as b moves has no effect on the total derivative of f w.r.t. b.

The envelope theorem for constrained maximization: Consider $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^1$ (differentiable) and

$\mathbf{h} : \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^m$ (differentiable). For $b \in \mathbb{R}$, suppose that $(\mathbf{x}^*(b), \boldsymbol{\lambda}^*(b))$ is the unique[1] vector satisfying the KKT conditions for the nonlinear programming problem:

$$\max_{\mathbf{x}} f(b, \mathbf{x}) \text{ such that } h^j(b, \mathbf{x}) \geq 0, \text{ for } j = 1, ..., m.$$

Then the total derivative of the function $f(b, \mathbf{x}^*(b))$ with respect to b is

$$\frac{df(b, \mathbf{x}^*(b))}{db} = \frac{\partial f(b, \mathbf{x}^*(b))}{\partial b} + \sum_{j=1}^{m} \lambda_j^*(b) \frac{\partial h^j(b, \mathbf{x}^*(b))}{\partial b}$$

Please note that the KKT for this more general version of the NPP is that

$$\bigtriangledown f(b, \mathbf{x}^*(b)) = -\textstyle\sum_{j=1}^{m} \lambda_j^*(b) \bigtriangledown h^j(b, \mathbf{x}^*(b)), \text{ with } \lambda_j^*(b) \geq 0 \text{ for all } j, \text{ and } h^j(b, \mathbf{x}^*(b)) > 0 \Longrightarrow \lambda_j^*(b) = 0.$$

[1] If there are multiple vectors satisfying the KKT conditions, then the maximized value of the objective need not be differentiable w.r.t. the parameters. For example, consider the problem $\max ||\mathbf{x}||$ s.t. $p_1 x_1 + p_2 x_2 = 1$. For $p_2 \geq 1 = p_1$, the maximized value of the objective is 1, attained when $\mathbf{x} = (1, 0)$; for $p_2 < 1$, the maximized value is $1/p_2$, attained when $\mathbf{x} = (0, 1/p_2)$. Thus the value function $M(p_2)$ has a kink at $p_2 = 1$.

In words, in this setting the gradient of f belongs to the *nonpositive* cone defined by the gradients of the constraints that are satisfied with equality at $\mathbf{x}^*(b)$). To reconcile this with our usual treatment, note that for the special case in which $h^j(b,\mathbf{x}) = b^j - g^j(\mathbf{x})$, we have $\bigtriangledown h^j = -\bigtriangledown g^j$, so that in this special case, the *nonnegative* cone defined by any subset of the gradients of g^j's is the *nonpositive* cone defined the corresponding subset of the gradients of h^j's.

Proof: The Lagrangian for this problem is

$$L(b,\mathbf{x}^*(b),\boldsymbol{\lambda}^*(b)) \quad = \quad f(b,\mathbf{x}^*(b)) \quad + \quad \sum_{j=1}^{m} \lambda_j^*(b) h^j(b,\mathbf{x}^*(b))$$

since $\lambda_j^*(b)h^j(b,\mathbf{x}^*(b)) = 0$ for all j, we have

$$M(b) \quad \equiv \quad f(b,\mathbf{x}^*(b)) \quad \equiv \quad L(b,\mathbf{x}^*(b),\boldsymbol{\lambda}^*(b))$$

It follows that

$$\begin{aligned}
\frac{dM(b,\mathbf{x}^*(b))}{db} &= \frac{df(b,\mathbf{x}^*(b))}{db} = \frac{dL(b,\mathbf{x}^*(b),\boldsymbol{\lambda}^*(b))}{db} \\
&= \frac{\partial L(b,\mathbf{x}^*(b),\boldsymbol{\lambda}^*(b))}{\partial b} + \sum_{i=1}^{n} \frac{\partial L(b,\mathbf{x}^*(b),\boldsymbol{\lambda}^*(b))}{\partial x_i}\frac{d{x^*}_i(b)}{db} + \sum_{j=1}^{m} \frac{\partial L(b,\mathbf{x}^*(b),\boldsymbol{\lambda}^*(b))}{\partial \lambda_j}\frac{d\lambda_j^*(b)}{db} \\
&= \frac{\partial f(b,\mathbf{x}^*(b))}{\partial b} + \sum_{j=1}^{m} \lambda_j^*(b)\frac{\partial h^j(b,\mathbf{x}^*(b))}{\partial b} \\
&\quad + \sum_{i=1}^{n}\left(\frac{\partial f(b,\mathbf{x}^*(b))}{\partial x_i} + \sum_{j=1}^{m}\lambda_j^*(b)\frac{\partial h^j(b,\mathbf{x}^*(b))}{\partial x_i}\right)\frac{dx_i(b)}{db} + \sum_{j=1}^{m} h^j(b,\mathbf{x}^*(b))\frac{d\lambda_j^*(b)}{db}
\end{aligned}$$

Since $\mathbf{x}^*(b)$ satisfies the KKT conditions each of the $\frac{\partial L(b,\mathbf{x}^*(b),\lambda^*(b))}{\partial x_i}$'s is zero. That is, for each i, the term in the large parentheses on the second line is zero. Moreover if $h(b,\mathbf{x}^*(b)) < 0$, then $\lambda_j^*(\cdot)$ is zero on a neighborhood of b and hence $\frac{d\lambda_j^*(b)}{db}$ is zero; Hence for all j $h^j(b,\mathbf{x}^*(b))\frac{d\lambda_j^*(b)}{db}$ is zero.

Conclude that all terms on the second line are zero. This proves

$$\frac{df(b,\mathbf{x}^*(b))}{db} = \frac{\partial f(b,\mathbf{x}^*(b))}{\partial b} + \sum_{j=1}^{m} \lambda_j^*(b) \frac{\partial h^j(b,\mathbf{x}^*(b))}{\partial b}$$

Here's an argument that "proves" the constrained envelope theorem in a slightly informal way. It is, however, *much* more informative than the formal proof above, and illuminates the reason why the movement in the $\mathbf{x}$ direction doesn't matter. Another virtue of the approach is that it uses the "mantra" form of the KKT, and doesn't involve the clumsy Lagrangian at all. To make the argument transparent, we consider the case of just one constraint, which we assume is binding. It's straightforward to extend it to the general case. We use the notation $\frac{d\mathbf{x}(b)}{db}db$ to denote the vector of derivatives of $\mathbf{x}(b)$ w.r.t. b. We have

$$M(b) \equiv f(b,\mathbf{x}^*(b))$$

so that, evaluating the differential at the point $\left(db, \frac{d\mathbf{x}(b)}{db}db\right)$

$$M(b+db) \approx \frac{\partial f(b,\mathbf{x}^*(b))}{\partial b}db + \underbrace{\bigtriangledown f_{\mathbf{x}}(b,\mathbf{x}^*(b))\frac{d\mathbf{x}(b)}{db}db}_{\text{Term A}} \tag{2}$$

where the notation $\bigtriangledown f_{\mathbf{x}}$ means the subvector of $\bigtriangledown f$ consisting of the partial derivatives of f w.r.t. the components of $\mathbf{x}$. Now the vector $\left(db, \frac{d\mathbf{x}(b)}{db}db\right)$ has the property that, to a first order approximation, a movement in this direction leaves the constraint function $h(\cdot,\cdot)$ unchanged at zero, i.e.,

$$h\left(b+db, \mathbf{x}^*(b) + \frac{d\mathbf{x}(b)}{db}db\right) - h(b,\mathbf{x}^*(b)) \approx \bigtriangledown h(b,\mathbf{x}^*(b)) \cdot \left(db, \frac{d\mathbf{x}(b)}{db}db\right) = 0.$$

or, rearranging

$$\frac{\partial h(b,\mathbf{x}^*(b))}{\partial b}db = -\bigtriangledown h_{\mathbf{x}}(b,\mathbf{x}^*(b)) \cdot \frac{d\mathbf{x}(b)}{db}db \tag{3}$$

but since the KKT conditions are satisfied

$$\bigtriangledown f_{\mathbf{x}}(b, \mathbf{x}^*(b)) \quad = \quad -\lambda(b) \bigtriangledown h_{\mathbf{x}}(b, \mathbf{x}^*(b)) \tag{4}$$

so that, multiplying both sides of (3) by $\lambda(b)$ and substituting from (4)

$$\lambda(b)\frac{\partial h(b, \mathbf{x}^*(b))}{\partial b}db \quad = \quad \underbrace{\bigtriangledown f_{\mathbf{x}}(b, \mathbf{x}^*(b))\frac{d\mathbf{x}(b)}{db}db}_{\text{Term A}}$$

We can now substitute for Term A in expression (2) to obtain

$$M(b + db) \quad \approx \quad \frac{\partial f(b, \mathbf{x}^*(b))}{\partial b}db + \lambda(b)\frac{\partial h(b, \mathbf{x}^*(b))}{\partial b}db$$

or, in derivative notation

$$\frac{dM(b)}{db} \quad = \quad \frac{\partial f(b, \mathbf{x}^*(b))}{\partial b} + \lambda(b)\frac{\partial h(b, \mathbf{x}^*(b))}{\partial b}$$

The great appeal of this argument is that *we never invoked the specific fact that* $\mathbf{dx} = \frac{d\mathbf{x}(b)}{db}db$, i.e., that the $\mathbf{x}$ vector changed in just the right direction so that f would be optimized at the new constraint, defined by $b + db$. The only property we used was that the change $\left(db, \frac{d\mathbf{x}(b)}{db}db\right)$ kept us on the level set of h corresponding to zero. In the above argument, we could have replaced the vector $\left(db, \frac{d\mathbf{x}(b)}{db}db\right)$ with *any* vector $(db, \mathbf{dx})$ with the property that $\bigtriangledown h(b, \mathbf{x}^*(b)) \cdot (db, \mathbf{dx}) = 0$, without changing the remainder of the proof in any way. The point being made here is the following: you start out at a (unique) point where the KKT conditions are satisfied, and the sole constraint is binding; you then shift the constraint by db; if you now move $\mathbf{x}$ in any direction whatsover, *so long as, to a first order approximation, you stay on the level set of* $h(\cdot, \cdot)$ *corresponding to zero*, i.e., choose some $\mathbf{dx}$ such that $\bigtriangledown h(b, \mathbf{x}^*(b))(db, \mathbf{dx}) = 0$, then, to a first order approximation, the change in $M(\cdot)$ will be the same. In other words, provided that, to a first order approximatino, you stay on the level set of $h(\cdot,\cdot)$ corresponding to zero, the direction you move in the $\mathbf{x}$ direction doesn't make any difference.

The reason why the direction in which you move $\mathbf{x}$ doesn't matter is quite transparent from the above argument: *any* vector $\mathbf{dx}$ that satisfies $\triangledown h(b, \mathbf{x}^*(b))(db, \mathbf{dx}) = 0$ necessarily satisfies $\triangledown h_{\mathbf{x}}(b, \mathbf{x}^*(b)) \cdot \mathbf{dx} = -\frac{\partial h}{\partial b} db$ as well. Because the KKT conditions are satisfied, this vector $\mathbf{dx}$ *must* increase f, to a first order approximation, by $\lambda(b)\frac{\partial h}{\partial b} db$. That is, all vectors $(b, \mathbf{x})$ that keep us on the zero level set of h must increase f (to a first order approximation) by exactly the same amount.

4.3. **Applications of the envelope theorem: Hotelling's and Shephard's lemmas.**

In the standard production problem, output and inputs are determined by solving an unconstrained optimization problem. The unconstrained envelope theorem provides an easy way to compute the supply function and input demand functions. (This is Hotelling's lemma.) Alternatively, if you are given the level of output, the constrained envelope theorem provides an easy way to compute the conditional input demand functions, where the conditioning is on the level of output. (This is Shephard's Lemma.) The development below draws very heavily on Varian's textbook, Microeconomic Analysis, which has been replaced by MWG, but is still very useful.

4.3.1. *Hotelling's Lemma.* Recall the general form of the unconstrained envelope theorem: we maximize $f(\mathbf{b}, \mathbf{x})$ w.r.t. $\mathbf{x}$ (no constraints). (In the above, I did the case where $\mathbf{b}$ and $\mathbf{x}$ were both scalars. Now we are considering vectors but the idea is exactly the same.) In this application $\mathbf{b} = (p, r, w)$, representing output price, rental rate and wage, and $\mathbf{x} = (k, \ell)$, representing capital and labor. Define

$$f(p, r, w; k, \ell) \quad \equiv \quad \Pi(p, r, w; k, \ell) \quad = \quad pq(k, \ell) - rk - w\ell$$

In our general treatment of the envelope theorem we defined $M(\mathbf{b}) = f(\mathbf{b}, \mathbf{x}(\mathbf{b}))$. The statement of the envelope theorem (unconstrained version) was that $\frac{dM}{db_i} = \frac{\partial M}{\partial b_i}$. In the present application,

$$M(p,r,w) \quad \equiv \quad \pi^*(p,r,w) \quad \equiv \quad pq(k^*,\ell^*) - rk^* - w\ell^*$$

where the vector (k^*,ℓ^*) solves the maximization problem for the vector of exog variables (p,r,w). (I should write $k^*(p,r,w)$, etc., but this would be too cumbersome. Just remember that the asterisk's indicate that now capital and labor are both *functions* of other variables, not scalars.) From the envelope theorem, we now have

(1) $\frac{dM(p,r,w)}{dp} \quad = \quad \frac{d\pi^*(p,r,w)}{dp} \quad = \quad \frac{\partial\pi^*(p,r,w)}{\partial p} \quad = \quad q(k^*,\ell^*).$

(2) $\frac{dM(p,r,w)}{dr} \quad = \quad \frac{d\pi^*(p,r,w)}{dr} \quad = \quad \frac{\partial\pi^*(p,r,w)}{\partial r} \quad = \quad -\, k^*.$

(3) $\frac{dM(p,r,w)}{dw} \quad = \quad \frac{d\pi^*(p,r,w)}{dw} \quad = \quad \frac{\partial\pi^*(p,r,w)}{\partial w} \quad = \quad -\, \ell^*.$

Note that we can look at these equations from two perspectives.

(1) They tell us how the value function changes as each of the exogenous variables change
(2) They also deliver, as a bonus, the supply function and input demand functions as the partial derivatives of the profit function.
   (a) Supply function: $q^*(p,r,w) \quad = \quad \frac{\partial\pi^*(p,r,w)}{\partial p}$.
   (b) Demand for capital: $k^*(p,r,w) \quad = \quad -\frac{\partial\pi^*(p,r,w)}{\partial r}$
   (c) Demand for labor: $\ell^*(p,r,w) \quad = \quad -\frac{\partial\pi^*(p,r,w)}{\partial w}$

   How do you actually use this Lemma? If you are given a closed from expression for the profit function, in terms of input and output prices, it would be very useful indeed: just take partial derivatives w.r.t. each prices. For example, suppose $f(\ell,k) = \sqrt{\ell^\alpha k^{1-\alpha}}$. Then

$$\pi^*(p,r,w) \quad \equiv \quad \sqrt{p\ell^{*\alpha}k^{*1-\alpha}} - rk^* - w\ell^* \quad = \quad 0.5p^2\left(\frac{w}{\alpha}\right)^{-2\alpha}\left(\frac{r}{1-\alpha}\right)^{-2(1-\alpha)},$$

   so taking derivatives is very easy.

On the other hand, if you start out just with the optimization problem, then you would have to get your explicit expressions for k^* and ℓ^* from the first order conditions of that problem before you had anything you could actually use.

4.3.2. *Shephard's Lemma.* Recall the general form of the constrained envelope theorem: we maximize $f(\mathbf{b},\mathbf{x})$ w.r.t. $\mathbf{x}$ subject to $h(\mathbf{b},\mathbf{x}) = 0$. In this application $\mathbf{b} = (\bar{q}, r, w)$ and $\mathbf{x} = (k, \ell)$, i.e., we are holding output $\bar{q}$ constant and minimizing cost. Define

$$f(r, w; k, \ell) \quad \equiv \quad C(r, w; k, \ell) \quad = \quad rk + w\ell$$

Our problem is now to minimize cost (maximize the negative of cost) s.t. $\bar{q} = q(k, \ell)$. In the general treatment we defined $M(\mathbf{b}) = f(\mathbf{b}, \mathbf{x}(\mathbf{b})) + \lambda h(\mathbf{b}, \mathbf{x}(\mathbf{b}))$. The statement of the envelope theorem (constrained version) was that $\frac{dM}{db_i} = \frac{\partial f}{\partial b_i} + \lambda \frac{\partial h}{\partial b_i}$. In the present application, the lagrangian for the maximization problem is

$$L(\bar{q}, r, w; k, \ell) \quad = \quad -(rk + w\ell) \quad + \quad \lambda(q(k, \ell) - \bar{q}).$$

Similarly, $M(\bar{q}, r, w) = -\hat{c}(\bar{q}, r, w) = -(r\hat{k} + w\hat{\ell})$, where the vector $(\hat{k}, \hat{\ell})$ solves the cost minimization problem for the vector of exog variables $(\bar{q}, r, w)$. From the constrained version of the envelope theorem, we have

$$(1) \quad \frac{dM(p,r,w)}{dr} \quad = \quad \frac{dL(\bar{q},r,w)}{dr} \quad = \quad -\frac{\partial \hat{c}(\bar{q},r,w)}{\partial r} \quad = \quad -\hat{k}.$$

$$(2) \quad \frac{dM(p,r,w)}{dw} \quad = \quad \frac{dL(\bar{q},r,w)}{dw} \quad = \quad -\frac{\partial \hat{c}(\bar{q},r,w)}{\partial w} \quad = \quad -\hat{\ell}.$$

Once again, we can look at these equations from two perspectives.

(1) They tell us how the cost function changes as two of the three exogenous variables change
(2) They also deliver, as a bonus, the *conditional* input demand functions as the partial derivatives of the *cost* function.

(a) Conditional demand for capital: $\hat{k}(\bar{q}, r, w) \quad = \quad \frac{\partial \hat{c}(\bar{q},r,w)}{\partial k}$

(b) Conditional demand for labor: $\hat{\ell}(\bar{q}, r, w) \quad = \quad \frac{\partial \hat{c}(\bar{q},r,w)}{\partial w}$

As before, this result is very useful *provided* you are given the specification of the cost function as a primitive. For example, consider the CES production function, $q(\ell, k) = [(\beta_k k)^\rho + (\beta_\ell \ell)^\rho]^{1/\rho}$. In this case, letting $\sigma = \rho(\rho - 1)$, the cost function is $\hat{c}(q^*, r, w) = \bar{q}\,[(r/\beta_k)^\sigma + (w/\beta_\ell)^\sigma]^{1/\sigma}$. We now have

$$\hat{k} \quad = \quad \frac{\partial \hat{c}(q^*, r, w)}{\partial r} \quad = \quad \bar{q}\,[(r/\beta_k)^\sigma + (w/\beta_\ell)^\sigma]^{1/\sigma - 1}\,(r/\beta_k)^\sigma / r.$$

4.4. **Another Application of the envelope theorem for constrained maximization**

Jacob Viner's famous figure of the long-run and short-run average total cost functions provides a nice example of the envelope theorem. Consider a production function $q = f(\ell, k)$. The LRATC curve assigns to each q the average total cost associated with the cost-minimizing combination of labor and capital. That is,

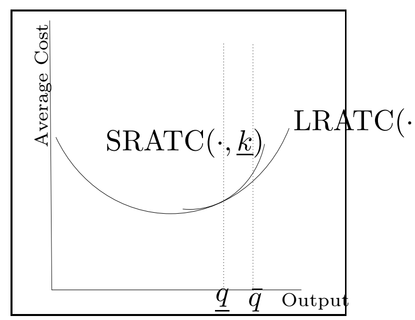


Figure 4. LRATC & SRATC

$$\text{LRATC}(q) \quad = \quad \min_{\{\ell,k\}} \frac{w\ell + rk}{q} \qquad \text{s.t.} \qquad q \geq f(\ell, k)$$

For each level of capital k, $\text{SRATC}^k(q)$ is the average cost of producing q using k and whatever is the required level of ℓ. For each q, $\text{LRATC}(q) = \text{SRATC}^k(q)$ at the level k that is optimal for that q. The graphs exhibit the well-known property that each SRATC curve is tangent to the LRATC curve at the point where they agree (see Fig. 4). Because of this relationship, the LRATC is referred to as the *outer envelope* of the SRATC curves. The "puzzle" here is that one would expect the short-run curve to be steeper than the long-run curve at $\bar{q}$, because when output increases from q, capital is held constant in the short-run, but varies in

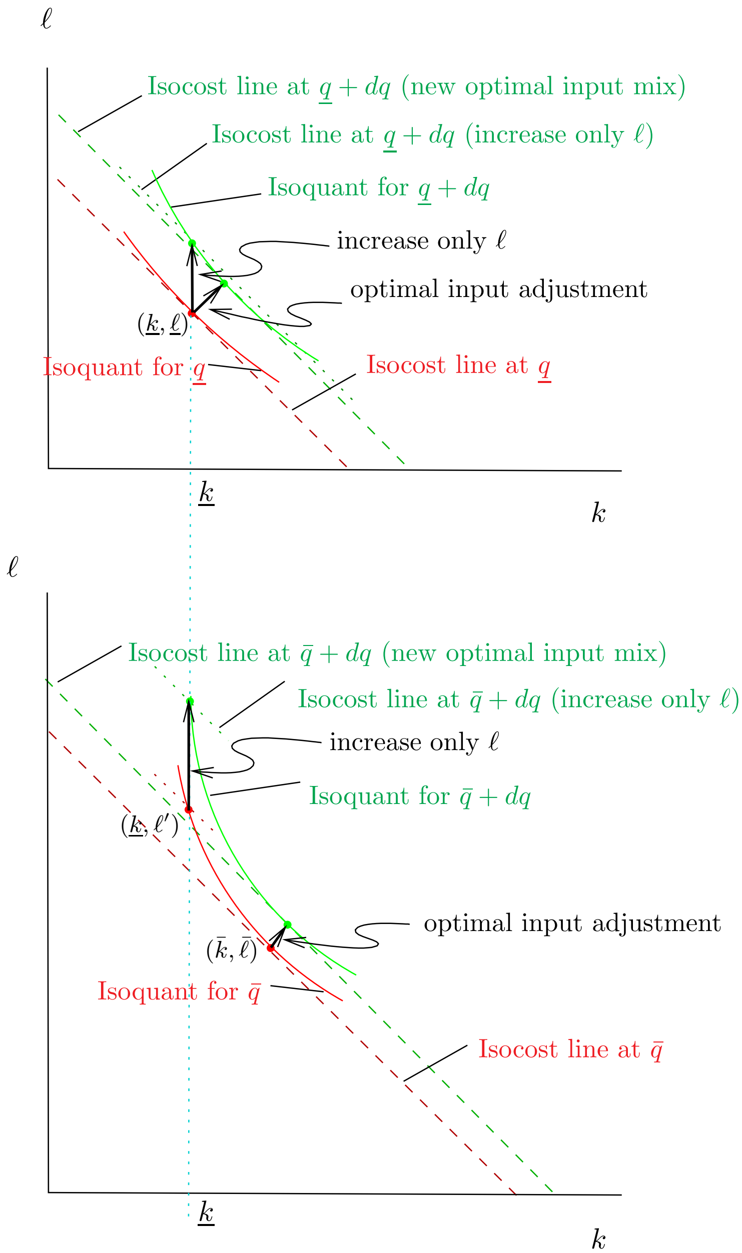


FIGURE 5. The envelope theorem applied to Viner's cost diagram.

the long run: we would expect that by adjusting both inputs in response to an increase in q, the producer could reduce costs relative to the case in which he is required to hold capital constant.

The envelope theorem resolves the puzzle. To see the relationship between this theorem and Fig. 4 we have to move into input space. See Fig. 5 which, note, is very similar to Fig. 3. Along the long-run average cost curve, we have solved the cost-minization problem, so that the constrained envelope theorem holds. For example, at the point $(\underline{q}, \text{LRATC}(\underline{q}))$ in Fig. 4, the input mix $(\underline{k}, \underline{\ell})$ is chosen to be at the point where the isoquant corresponding to q is tangent to the iso-cost line. Now suppose as in the upper panel of Fig. 5, you move from $(\underline{k}, \underline{\ell})$ to the new isoquant line corresponding to $\underline{q} + dq$. To a first order approximation, the new isoquant will necessarily be parallel to the old (because, again, the gradient is constant in a first order approximation). As the figure illustrates, it doesn't matter much which way you combine inputs in order to get to the new isoquant: to a first order approximation, all input mixes result in the same cost increment. In particular, if you produce the additional output entirely by increasing labor (as you do along the SRATC curve), then the increment in your cost is, to a first order approximation the same as if you had increased both inputs in the optimal proportions (as you do along the LRATC curve). In other words, the *slopes* of your LRATC and SRATC curves are the same at $\underline{q}$.

Now suppose you are producing a higher level of output, $\bar{q}$. In the bottom panel, we compare the case in which we are producing with the original, now suboptimal level of capital $\underline{k}$ and labor $\ell' > \bar{\ell}$, against the level of inputs $(\bar{k}, \bar{\ell})$ that would be optimal for this new higher level $\bar{q}$. This situation is represented by the lower panel of Fig. 5. As the figure indicates, when you move from the isoquant corresponding to $(\bar{k}, \bar{\ell})$ to the new isoquant line corresponding to $\bar{q} + dq$, it matters a lot whether you are starting from the optimal input mix $(\bar{k}, \bar{\ell})$ or starting from your *original* capital level $\underline{k}$, which requires an input mix of $(\underline{k}, \ell')$. There are two things to notice:

(1) Notice that when you produce with $(\bar{k}, \bar{\ell})$ you are on a lower isocost line than when you produce with $(\underline{k}, \ell')$. This observation corresponds to the fact that in the Average Cost diagram (Fig. 4) above the point $\bar{q}$ the *level* of the short run average cost curve is higher than the *level* of the long run average cost curve.

(2) Now observe that if you are initially producing $\bar{q}$ and then you shift to producing $\bar{q} + dq$, if your starting point is the suboptimal mix $(\underline{k}, \ell')$, and you are allowed *only* to increase ℓ because k is fixed, then the increase in costs is *much* larger than if your starting point were the optimal mix $(\bar{k}, \bar{\ell})$, and you adjust both inputs. This observation corresponds to the fact that in the Average Cost diagram (Fig. 4) above the point $\bar{q}$ the *slope* of the short run average cost curve is steeper than the *slope* of the long run average cost curve.

Here's the formal math. Let $\big(\ell(q), k(q)\big)$ denote the optimal input mix for producing a given level q. We want to use the envelope theorem to prove that the following result:

$$\frac{d\text{LRATC(q)}}{dq} = \frac{d\text{SRATC(q,k(q))}}{dq}.$$

The long run average cost is the solution to the following constrained problem

$$\max_{\ell,k} \big(-\text{TC}(\ell,k)/q \big) \quad \text{s.t.} \quad f(\ell,k) \geq q$$

The short run average cost is the solution to the same problem, with the addition of one more constraint

$$\max_{\ell,k} \big(-\text{TC}(\ell,k)/q \big) \quad \text{s.t.} \quad f(\ell,k) \geq q \text{ and } k = k^*,$$

The Lagrangian for the long-run problem is

$$L^{\text{LR}} = -\left(\frac{w\ell + rk}{q}\right) + \lambda_1\big(f(\ell,k) - q\big)$$

Let $(\ell^{\text{LR}}, k^{\text{LR}}, \lambda_1^{\text{LR}})$ be the solution to this problem. Note for future reference that

$$0 = \frac{dL^{\text{LR}}}{dk} = -\frac{\partial}{\partial k}\left(\frac{w\ell^{\text{LR}} + rk^{\text{LR}}}{q}\right) + \lambda_1^{\text{LR}} \frac{\partial f(\ell^{\text{LR}}, k^{\text{LR}})}{\partial k} \tag{5}$$

By the envelope theorem, we know that

$$\frac{d\text{LRATC}(q)}{dq} \quad = \quad \lambda_1^{\text{LR}} \quad - \quad \frac{\partial}{\partial q}\left(\frac{w\ell^{\text{LR}} + rk^{\text{LR}}}{q}\right)$$

Now let $k^* = k^{\text{LR}}$. The Lagrangian for the short-run problem is

$$L^{\text{SR},k^{\text{LR}}} \quad = \quad -\frac{w\ell + rk}{q} \quad + \quad \mu_1\big(f(\ell,k) - q\big) \quad + \quad \mu_2\big(k - k^{\text{LR}}\big)$$

Let $(\ell^{\text{SR}}, k^{\text{SR}}, \mu_1^{\text{SR}}, \mu_2^{\text{SR}})$ be the solution to this problem. Note for future reference that

$$0 \quad = \quad \frac{dL^{\text{SR},k^{\text{LR}}}}{dk} \quad = \quad -\frac{\partial}{\partial k}\left(\frac{w\ell^{\text{SR}} + rk^{\text{SR}}}{q}\right) \quad + \quad \mu_1^{\text{SR}}\frac{\partial f(\ell^{\text{SR}}, k^{\text{SR}})}{\partial k} \quad - \quad \mu_2^{\text{SR}} \tag{6}$$

Now since $k^{\text{SR}} = k^{\text{LR}}$, it follows that $\ell^{\text{SR}} = \ell^{\text{LR}}$. Moreover, since $(\ell^{\text{LR}}, k^{\text{LR}})$ is the optimal input mix for producing q, it follows that $\mu_2^{\text{SR}} = 0$ (i.e., you can't improve the objective function by either increasing or decreasing the constant in the second constraint. That is, in the solution this constraint is satisfied with equality but is not *binding*.) But if $\mu_2^{\text{SR}} = 0$, then, comparing expressions (5) and (6), it follows that $\mu_1^{\text{SR}} = \lambda_1^{\text{LR}}$. Finally, by the envelope theorem, we know that

$$\begin{aligned} \frac{d\text{SRATC}(q)}{dq} \quad &= \quad \mu_1^{\text{SR}} \quad - \quad \frac{\partial}{\partial q}\left(\frac{w\ell^{\text{SR}} + rk^{\text{SR}}}{q}\right) \\ &= \quad \lambda_1^{\text{LR}} \quad - \quad \frac{\partial}{\partial q}\left(\frac{w\ell^{\text{LR}} + rk^{\text{LR}}}{q}\right) \\ &= \quad \frac{d\text{LRATC}(q)}{dq} \end{aligned}$$