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5. CHARACTERISTICS OF FUNCTIONS.

5.1. Surjective, Injective and Bijective functions

Definition: A function $f : X \rightarrow A$ is *onto* or *surjective* if every point in the range is reached by f starting from some point in the domain, i.e., if $\forall a \in A, \exists x \in X$ such that $f(x) = a$.

Example: consider $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 5$ for all x . This function is *not* surjective because there are points in the range (i.e., \mathbb{R}) that aren't reachable by f (i.e., any point in \mathbb{R} except 5 is unreachable).

Definition: A function $f : X \rightarrow A$ is *1-1* or *injective* if for every point in the range, there is *at most* one point in the domain that gets mapped by f to that point. More precisely, f is injective if for all x, x' such that $x \neq x', f(x) \neq f(x')$.

Example: consider $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 5$ for all x . This function is *not* injective because there is a point in the range (i.e., 5) that gets mapped to by f , starting from a large number of points in the domain.

Definition: A function $f : X \rightarrow A$ is *bijective* if it is surjective and injective.

Definition: A function $f : X \rightarrow A$ is *invertible* if there exists a function $f^{-1} : A \rightarrow X$ such that for all $a \in A$, $f(f^{-1}(a)) = a$ and for all $x \in X$, $f^{-1}(f(x)) = x$. (We'll show below that you need *both* of these conditions to capture the notion of invertibility.) If such a function exists, it is called the *inverse* of f .

Theorem: A function $f : X \rightarrow A$ is *invertible* if and only if it is bijective.

- You need f to be surjective (onto): otherwise there would be points in the range that aren't mapped to by f from any point in the domain: E.g., take $f : [0, 1] \rightarrow \mathbb{R}$ defined by $f(x) = 5$: the image of $[0, 1]$ under f is $\{5\}$ Now consider any point $a \notin \{5\}$. Since f doesn't map anything to a , there cannot exist a function f^{-1} such that $f(f^{-1}(a)) = a$.
- You need f to be injective (1-1): otherwise the inverse couldn't be a function. E.g., take $f : [0, 1] \rightarrow [0, 1/4]$ defined by $f(x) = x - x^2$ and take $a = 3/16$; this point is reached from both $1/4$ and $3/4$, i.e., $f(1/4) = f(3/4) = 3/16$. Now apply the definition to both $1/4$ and $3/4$: it is required that for all $x \in X$, $f^{-1}(f(x)) = x$. In particular, it is required that $f^{-1}(f(1/4)) = f^{-1}(3/16) = 1/4$. But it is also required that $f^{-1}(f(3/4)) = f^{-1}(3/16) = 3/4$. Thus f is required to map the same point to two different places, and hence can't be a function.
- In the definition of invertibility, we require that
 - (i) for all $a \in A$, $f(f^{-1}(a)) = a$ and
 - (ii) for all $x \in X$, $f^{-1}(f(x)) = x$?

Why wouldn't just one of these be sufficient? To see why not, consider the following:

- (1) $f : [0, 1] \rightarrow [0, 2]$ defined by $f(x) = x$; Here $X = [0, 1]$ and $A = [0, 2]$. Note that f is not onto. Now consider a candidate $f^{-1} : A \rightarrow X$ defined by $f^{-1}(a) = \begin{cases} a & \text{if } a \leq 1 \\ 0 & \text{if } a \in (1, 2] \end{cases}$.

Clearly this is not the true inverse of f ! However, for all $x \in X = [0, 1]$, $f^{-1}(f(x)) = f^{-1}(x) = x$, so requirement (ii) above is indeed satisfied. Fortunately, however, requirement (i) is not satisfied, for all $a \in (1, 2]$. To see this note that $f(f^{-1}(a)) = f(0) = 0 \neq a$.

- (2) (This is essentially Susan's example. Thanks, Susan, for figuring it out.) $f : [0, 2] \rightarrow [0, 1]$ defined by $f(x) = \begin{cases} x & \text{if } x \leq 1 \\ 2 - x & \text{if } x \in (1, 2] \end{cases}$. Here $X = [0, 2]$ and $A = [0, 1]$. Now consider a candidate $f^{-1} : A \rightarrow X$ defined by $f^{-1}(a) = a$. Note that it is perfectly legal to say that X is the range of f^{-1} , it's just that f^{-1} isn't onto. In this case, for all $a \in A = [0, 1]$, $f(f^{-1}(a)) = f(f(a)) = a$. So requirement (i) above is satisfied. Fortunately, however, requirement (ii) is not satisfied, for all $x \in (1, 2]$. To see this note that $f^{-1}(f(x)) = f^{-1}(2 - x) = 2 - x \neq x$.

These two examples establish that you need conditions (i) and (ii) to characterize the intuitive definition of invertibility.

5.2. Homotheticity

Defn: A function is homothetic if its level sets are simply magnifications of each other.

More precisely, a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is homothetic if for all \mathbf{x}, \mathbf{y} such that $f(\mathbf{x}) = f(\mathbf{y})$ and all $k \in \mathbb{R}$, $f(k\mathbf{x}) = f(k\mathbf{y})$.

Another characterization is that supporting hyperplanes along a ray are all parallel

Examples: Cobb-Douglas; CES production and utility functions.

Cf. draw picture with indifference curves not homothetic.

5.3. Homogeneity and Euler's theorem

Defn: A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is homogeneous of degree k if for every \mathbf{x} and every scalar α , $f(\alpha\mathbf{x}) = \alpha^k f(\mathbf{x})$.

Examples: $u(\mathbf{x}) = \sqrt{x_1 x_2}$ is homogeneous of degree 1; $u(\mathbf{x}) = x_1 x_2$ is homogeneous of degree 2.

Relationship between homotheticity and homogeneity: any homogeneous function is homothetic but not necessarily the other way around.

Note that *any* homogeneous of degree 1 function delivers a *linear* function when you take cross-section that passes thru the origin, i.e., cut the graph along a line thru the origin.

Th'm (Euler): Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is homogeneous of degree k ; then for all x ,

$$f_1(\mathbf{x})x_1 + \dots + f_n(\mathbf{x})x_n = kf(\mathbf{x})$$

Proof is in Silberberg (p. 91)

In particular, if homogeneous of degree 1, then

$$f_1(\mathbf{x})x_1 + \dots + f_n(\mathbf{x})x_n = f(\mathbf{x})$$

Adding up condition of constant returns to scale production function: f is a production function, $f_i(\mathbf{x})$ is the marginal product of factor i ; if each factor is paid its marginal product, then nothing left over; more generally, if the production function exhibits *increasing* returns to scale (i.e., is homogeneous of degree $k > 1$), then paying each factor its marginal product would *more than* exhaust all of the output; if the production function exhibits *decreasing* then after paying factors their marginal products, there will be surplus product left over.

Another example is that demand functions tend to be homogeneous of degree zero in prices and income: i.e., double all of these things, and no change in demand.

5.4. Monotonic functions

A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is monotonic if $y > x$ implies $f(y) > f(x)$; similarly for $f : \mathbb{R}^n \rightarrow \mathbb{R}$, but there are variations depending on whether $y > x$ means in every component or just one.

5.5. Concave and quasi-concave functions; Definiteness, Hessians and Bordered Hessians.

Definiteness

Defn: A matrix $A(n \times n)$ is positive definite if for every $x \in \mathbb{R}^n$, $x'Ax > 0$.

That is, the angle between x and Ax is acute, for every x ; that is, the matrix A moves every x by no more than 90 degrees.

Defn: A matrix $A(n \times n)$ is negative definite if for every $x \in \mathbb{R}^n$, $x'Ax < 0$.

That is, the angle between x and Ax is obtuse, for every x ; that is, the matrix A moves every x by more than 90 degrees.

Defn: A matrix $A(n \times n)$ is positive semidefinite if for every $x \in \mathbb{R}^n$, $x'Ax \geq 0$.

That is, the angle between x and Ax is acute or 90 degrees, for every x ; that is, the matrix A moves every x by no more than 90 degrees.

Defn: A matrix $A(n \times n)$ is negative semidefinite if for every $x \in \mathbb{R}^n$, $x'Ax \leq 0$.

That is, the angle between x and Ax is obtuse or 90 degrees, for every x ; that is, the matrix A moves every x by 90 degrees or more

Defn: A matrix $A(n \times n)$ is indefinite if for *some* $x, y \in \mathbb{R}^n$, $x'Ax > 0$ and $y'Ay < 0$.

That is, the angle between x and Ax is acute while the angle between y and Ay is obtuse.

The following completely mindless way of testing for definiteness of *symmetric* matrices involves principal minors. (The following is very brief. For the excruciating details, see Simon and Blume, pp 381-383)

Defn: the k 'th leading principal minor of a matrix is the determinant of the top left-hand corner $k \times k$ submatrix. (A nonleading principal minor is obtained by deleting some rows and *the same* columns from the matrix, e.g., delete rows 1 and 3 and columns 1 and 3.)

- A symmetric matrix A is positive definite iff the leading principal minor determinants are all positive
- A symmetric matrix A is negative definite iff the leading principal minor determinants alternate in sign, starting with negative, i.e., sign is $(-1)^k$ for k 'th principal minor.
- A symmetric matrix A is positive semidefinite iff *all* of the principal minor determinants are nonnegative (not just the leading ones).
- A symmetric matrix A is negative semidefinite iff *all* of the *nonzero* k 'th principal minor determinants (not just the leading ones) have the same sign as $(-1)^k$.

The difference between the tests for definiteness and for semi-definiteness is that when you test for semi-definiteness, one of the *leading* principal minors could be zero.

Example:

- Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ be the identity matrix. This is clearly positive definite. Check minors: 1 and 1-0. All positive
- Let $B = -A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. be the negative of the identity matrix. This is clearly negative definite. Check minors: -1 and 1-0. Alternate, starting negative.
- Let $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ be a identity matrix with the rows permuted. This is indefinite. Eigenvectors are the 45 degree line, which stays put (eigenvalue is 1) and the negative 45 degree line, which gets flipped (eigenvalue is -1). Check minors: 0 and 0-1. Second minor is negative. fails all of the conditions above.
- Let $D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. First suppose that $a = 0$. This case illustrates why if the first leading principal minor is zero, then you have to look at the other first principal minor. It's easy to check that
 - if $b > 0$, then D positive semidefinite;
 - if $b < 0$, then D negative semidefinite;

The surprising thing is that if a is tiny but small, it's enough to just check a , and you don't have to worry about checking the nonleading first principal minor. This example is simple enough that you can see how things work. Suppose $a > 0$

- if $b > 0$, then $\det(D)$ is positive and the conditions for positive definiteness are satisfied.
- if $b < 0$, then $\det(D)$ is negative and D fails all of the definiteness tests (so no point in checking the nonleading first principal minor.)

As an exercise, check to see why you don't have to check the nonleading first principal minor if $a < 0$.

Defn: a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is concave if the set $\{(\mathbf{x}, y) \in \mathbb{R}^n \times \mathbb{R} : y \leq f(\mathbf{x})\}$ is a convex set; that is, for every $\mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^n$, and $y^1 \leq f(\mathbf{x}^1), y^2 \leq f(\mathbf{x}^2) \in \mathbb{R}^n$, the line segment joining (\mathbf{x}^1, y^1) and (\mathbf{x}^2, y^2) lies underneath the graph of f .

Theorem: a twice continuously differentiable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is concave (strictly concave) iff for every \mathbf{x} ,

$$f(\bar{\mathbf{x}} + \mathbf{dx}) - f(\bar{\mathbf{x}}) \leq \quad (<) \quad \nabla f(\bar{\mathbf{x}}) \cdot \mathbf{dx}. \quad (1)$$

Notice that there isn't a qualification on the size of \mathbf{dx} .

The unparenthesized part of the preceding theorem is equivalent to the following statement: a twice continuously differentiable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is concave iff for every $x \in \mathbb{R}^n$, the tangent plane to f at x lies *everywhere* weakly above the graph. Recall what we did with Taylor's theorem earlier.

Theorem: a twice continuously differentiable function f is concave iff for every \mathbf{x} , the Hessian of f at \mathbf{x} is negative semi-definite. On the other hand, $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is strictly concave if for every \mathbf{x} the Hessian of f at \mathbf{x} is negative definite.

Note that the condition for strict concavity is only sufficient. The usual counter-example establishes it is not necessary: $f(x) = -x^4$ is strictly concave, but at $x = 0$ the Hessian is not negative definite. On the other hand, the condition for weak concavity is both necessary and sufficient. To see this, recall from the Taylor-Lagrange theorem that for some $\lambda \in [0, 1]$,

$$\begin{aligned} f(\bar{\mathbf{x}} + \mathbf{dx}) - f(\bar{\mathbf{x}}) &= \nabla f(\bar{\mathbf{x}}) \cdot \mathbf{dx} + \mathbf{dx}' H f(\bar{\mathbf{x}} + \lambda \mathbf{dx}) \mathbf{dx} \\ &\leq \nabla f(\bar{\mathbf{x}}) \cdot \mathbf{dx} \quad \text{iff} \quad \mathbf{dx}' H f(\bar{\mathbf{x}} + \lambda \mathbf{dx}) \mathbf{dx} \leq 0 \end{aligned}$$

Reverse everything for convex functions.

From what we saw about second order conditions above, you should be able to see now why economists always assume that the functions they are interested in are concave (e.g., profit functions) when they want to maximize them, and convex (e.g., cost functions) when they want to minimize them.

Important to emphasize that concavity buys you much more than second order conditions can *ever* buy you.

- calculus inherently can only give you local information, any more than a cat can talk.
- concavity is a global property of the function

Concavity guarantees you GLOBAL conditions.

Concavity is a very strong condition to impose in order to guarantee existence of unique maxima, etc. Turns out that you can make do with a condition that is much weaker, i.e., *quasi-concavity*, provided you are doing *constrained* optimization on a convex set. Specifically, it is almost but *definitely not exactly* true that if $f : X \rightarrow A$ is *strictly quasi-concave* and X is convex, then f attains a *unique global* maximum on X at $\bar{\mathbf{x}}$ if and only if the first order condition (Karush-Kuhn-Tucker/Mantra) is satisfied at $\bar{\mathbf{x}}$ (There are in fact caveats in each direction of this relationship, that's why it's not exactly true—for example, consider $f(x) = x^3$ on the interval $[-1, 1]$; KKT is satisfied at $x = 0$, but $f(\cdot)$ is not maximized on $[-1, 1]$ at zero—but I want you to get the basic idea.)

It is not true that if f is quasi-concave, the gradient is zero is necessary and sufficient for a maximum. Example, $f(x) = x^3$ is monotone and hence quasi-concave. It's gradient is zero at $x = 0$ but this isn't a maximum.

Defn: a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is quasiconcave if for every $\alpha \in \mathbb{R}$, the set $\{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \geq \alpha\}$ is a convex set; that is, if all of the upper contour sets are all convex.

Replace upper with lower for quasiconvex.

Theorem (SQC): A sufficient condition for f to be strictly quasi-concave is that for all \mathbf{x} and all $d\mathbf{x}$ such that $\nabla f(\bar{\mathbf{x}})d\mathbf{x} = 0$, $d\mathbf{x}'Hf(\bar{\mathbf{x}})d\mathbf{x} < 0$.

Bordered Hessian

For checking for quasi-concavity, use something called the bordered Hessian of f :

$$\begin{bmatrix} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{bmatrix}$$

Defn: the k 'th leading principal minor of a *bordered* matrix is the determinant of the top left-hand corner $k + 1 \times k + 1$ submatrix (i.e., the border is always included). The defn for a nonleading principal minor is the same as for an unbordered matrix, except that the border is included.

Theorem: A twice continuously differentiable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that ∇f is never zero is strictly quasi-concave iff and only for every $\mathbf{x} \in \mathbb{R}^n$, the *second* through n 'th leading principal minors of the Bordered Hessian of f , evaluated at \mathbf{x} , are alternatively positive and negative in sign; Moreover, if f is quasi-concave, then for every $\mathbf{x} \in \mathbb{R}^n$, the *second* through n 'th (not necessarily leading) principal minors of the Bordered Hessian of f , evaluated at \mathbf{x} , are alternatively nonnegative and nonpositive in sign.

Whatever happened to the *first* principal minor. It's *always* $-f_1^2 < 0$, so there is nothing to check.

Theorem: $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is strictly quasi-convex iff $-f$ is strictly quasi-concave.

Example: Let $f : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ be defined by $f = x_1x_2$. Note that the domain of this function is the positive orthant. Gives Cobb-Douglas indifference curves, so we know it is quasi-concave, but if you look along rays through the origin you get convex (quadratic) functions. We'll check that it satisfies the minor test: $\nabla f = (x_2, x_1)$ and $Hf = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which, as we have seen is indefinite. However, the bordered hessian is

$$BHf = \begin{bmatrix} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{bmatrix} = \begin{bmatrix} 0 & x_2 & x_1 \\ x_2 & 0 & 1 \\ x_1 & 1 & 0 \end{bmatrix}$$

The first principal minor is $-x_2^2 < 0$. Expanding by cofactors, the second principal minor is $2x_1x_2 \geq 0$, which is certainly nonnegative on the nonnegative orthant, so this satisfies the test for (weak) quasi-concavity.