ARE211, Fall 2009

ANALYSIS6: TUE, SEP 15, 2009 PRINTED: SEPTEMBER 29, 2009 (LEC# 6)

Contents

1. Analysis (cont)

1.9. **Continuous Functions**

A function is continuous if it maps nearby points to nearby points. Draw the graph without taking pen off paper. Graph is connected. Formally:

Definition: Consider $\mathbf{f} : X \to \mathbb{R}^k$. Fix $\mathbf{x}_0 \in X$. The function $\mathbf{f}$ is *continuous at* $\mathbf{x}_0$ if whenever $\{\mathbf{x}_m\}_{m=1}^{\infty}$ is a sequence in X which converges to $\mathbf{x}_0$, then $\{\mathbf{f}(\mathbf{x}_m)\}_{m=1}^{\infty}$ converges to $\mathbf{f}(\mathbf{x}_0)$. The function $\mathbf{f}$ is *continuous* if it is continuous at $\mathbf{x}$, for every $\mathbf{x} \in X$.

While this definition seems obvious and intuitive, all is not what it seems to be. We've noticed ad nauseum that whether or not a sequence converges depends on the metric and the universe. In this section, we now have *two* different sequences, one in the domain and one in the range; so we have to worry about what kinds of things converge in the domain and what kinds of things converge in the range. So we have roughly double the number of counter-intuitive possibilities. To see what kinds of things can happen, consider the following question:

Question: let $\mathbf{f} : X \to \mathbb{R}^k$ and let X be endowed with the discrete metric. What can we say about the continuity of $\mathbf{f}$?

Answer: $\mathbf{f}$ is continuous.

In other words, if X is endowed with the discrete metric, then *every* function is continuous! The reason for this is that the discrete metric makes convergence an *extremely* stringent requirement: (x_n) converges to x, iff, eventually, the sequence is constant at x. But whenever this stringent condition is satisfied, i.e., we have a constant sequence in the domain, the resulting sequence in the range, $\{f(x_n)\}$ is also, trivially, constant. That is, convergence in the domain is *so* difficult to accomplish that whenever it is accomplished, convergence in the range is assured.

Now consider the reverse question:

Question: Let $\mathbf{f} : X \to \mathbb{R}^k$, where $\mathbb{R}^k$ is endowed with the discrete metric. What can we say about the continuity of $\mathbf{f}$.

Answer: (first attempt). An obvious first shot at an answer to this question is: $\mathbf{f}$ is continuous iff it is constant. The "if" part is right but the "only if" part is very wrong.

(1) It matters what the metric is on X. If the metric on X is the discrete metric, then, as we just saw, *every* function is continuous.

(2) Now let's assume that the metric on X is the Euclidean metric and restrict attention to functions that map X to $\mathbb{R}^1$. In this case, there are non-constant functions which *are* continuous, provided that the domain has "holes" in it. For example,

- if the space X is a discrete space in the Euclidean metric (e.g., if X is the integers), then, obviously, every function $f : X \to \mathbb{R}$ is continuous
- consider the function $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases} \tag{1}$$

  In this case, there's a "hole" in the domain at zero and we don't have to worry about sequences that *would* converge to zero if the domain were $\mathbb{R}$.
- Finally, let $X = \{1/n : n \in \mathbb{N}\}$ and consider the "function" $f : X \to \mathbb{R}$ defined by $f(1/n) = (-1)^n/n$. This function has the apparently discontinuous property that the corresponding sequence in the range, $\{f(1/n)\}$, bounces up and down between 1 and -1. It *looks* like this sequence in the range ought to fail the requirement that the definition of continuity imposes, even with the *regular* metric on the range of the function, and most certainly with the discrete metric on the range. *But*, since $0 \notin X$, the

sequence $\{1/n\}$ doesn't converge in X—indeed, it doesn't even have any subsequences that converge—so the definition of continuity imposes absolutely no restriction on any sequence in the range.

Answer: (second (and correct) attempt). If $\mathbf{f} : X \to \mathbb{R}^k$, where X is endowed with a Euclidean metric and $\mathbb{R}^k$ is endowed with the discrete metric, then $\mathbf{f}$ is continuous iff $\mathbf{f}$ is constant *on every connected subset* of X. I haven't defined connected yet, but, intuitively, a set is connected if it is, well, connected. More precisely, a set $C \subset \mathbb{R}$ is connected if for any $x, y \in C$, the entire line segment connecting x to y is also in C. In the example given by display (1) above, the domain X has two connected subsets, i.e., $\mathbb{R}_{++}$ and $\mathbb{R}_{--}$.

1.9.1. *Continuous functions on compact sets attain extreme values.* For us, the most important result relating to continuity is the following theorem, sometimes know as the "extreme value theorem."

Theorem: (Weierstrass) Consider a function $f : X \to \mathbb{R}^1$, where both X and $\mathbb{R}$ are endowed with the Euclidean metric. If X is a compact set and f is continuous on X, then f attains a global maximum and a global minimum on X.

Intuitive sketch of the proof:

- show that if f is continuous and X is compact, then the image of the function must be bounded. (We'll prove this in a separate lemma.)
- let $\bar{f}$ denote the supremum of the image of the function. Since f is bounded we know that the supremum is a real number, not infinity. Our task now is to show that this supremum is actually attained, and is hence the maximum value of the function.
- pick a sequence $\{x_n\}$ such that the sequence $\{f(x_n)\}$ gets closer to the supremum.

- while the sequence $\{x_n\}$ needn't converge, it follows from the compactness of X that there must exist a subsequence $\{y_n\}$ of $\{x_n\}$ such that $\{y_n\}$ converges to $y \in X$.
- since f is continuous, $\{f(y_n)\}$ must converge to $f(y)$. But by our choice of the sequence $\{x_n\}$, of which $\{f(y_n)\}$ is a subsequence, and by defn of the supremum, $\{f(y_n)\}$ converges also to $\bar{f}$. Hence $f(y) = \bar{f}$.
- Since $\bar{f}$ is the supremum of the set $f(X) = \{f(x) : x \in X\}$, then $f(y) = \bar{f} \geq f(x)$, for all $x \in X$.

Before proceeding to the formal proof, we'll prove a completely obvious Fact about sequences and subsequences. (We could call it a Lemma, but that's glorifying it.) Recall that to show that (t_n) is a subsequence of (x_n), you always have to show the existence of a *strictly increasing* function $\tau : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, $t_n = x_{\tau(n)}$. Now note:

Fact: : If $\tau : \mathbb{N} \to \mathbb{N}$ is a strictly increasing mapping, then for all n, $\tau(n) \geq n$.

To prove this, we'll argue by induction.

- *Initial step:* $\tau(1) \geq 1$ (duh).
- *Inductive step:* suppose that $\tau(n) \geq n$; then $\tau(n+1) \geq n+1$.

  Proof of the Inductive step: Let $\tau(n) = k \in \mathbb{N}$, $k \geq n$. since $\tau(n+1) > \tau(n)$ and $\tau(n+1) \in \mathbb{N}$, $\tau(n+1) \geq k+1 \geq n+1$.

Now for the lemma establishing that continuity of the function plus compactness of the domain implies boundedness.

Lemma: Consider $f : X \to \mathbb{R}$, where both X and $\mathbb{R}$ are endowed with the Euclidean metric. If f is continuous and X is compact, then f is bounded.

Proof of the Lemma: We'll just show that the function is bounded above, by proving that if X is compact and f isn't bounded, then f cannot be continuous. Assume that f isn't bounded, i.e., for all $n \in \mathbb{N}$, $\exists x \in X$ such that $f(x) > n$. In other words, we can build a sequence $\{x_n\}$ in the following way: for each $n \in \mathbb{N}$, choose some point in the domain such that f of this point exceeds n; call that point x_n. Since X is compact, the sequence $\{x_n\}$ contains a convergent subsequence. Call this subsequence $\{y_n\}$ and let $y \in X$ denote its limit. Define $\tau : \mathbb{N} \to \mathbb{N}$ by $y_n = x_{\tau(n)}$ for all n. Since f is defined on X, $f(y) \in \mathbb{R}$, that is, $f(y) < N$, for some $N \in \mathbb{N}$. Now pick $n \geq N + 1$ and note that by the Fact above, $\tau(n) \geq n \geq N + 1$. Moreover, by assumption, $f(y_n) = f(x_{\tau(n)}) \geq \tau(n) \geq N + 1$. Hence for all $n > N + 1$, $f(y_n) - f(y) > 1$, so that f is not continuous at y. Similarly, f is bounded below.

Proof of the Theorem: Let $\bar{f}$ denote the supremum of the image of X under f, i.e., the set $\{f(x) : x \in X\}$. By the Lemma above, $\bar{f} \in \mathbb{R}$, i.e., $\bar{f}$ is the least upper bound for the set $f(X)$. By Theorem B from a few lectures ago, for all n, there exists x_n such that $f(x_n) > \bar{f} - 1/n$. Since X is compact, the sequence $\{x_n\}$ contains a convergent subsequence. Call this subsequence $\{y_n\}$ and let $y \in X$ denote its limit. Since f is continuous, the sequence $\{f(y_n)\}$ converges to $f(y) \in \mathbb{R}$.

To complete the proof, we'll show that $\{f(y_n)\}$ also converges to $\bar{f}$. Since $\{y_n\}$ is a subsequence of $\{x_n\}$, there exists a strictly increasing mapping $\tau : \mathbb{N} \to \mathbb{N}$ such that for all n, $y_n = x_{\tau(n)}$. To prove that $\{f(y_n)\}$ converges to $\bar{f}$, we need to show that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $f(y_n) \in B(\bar{f}, \epsilon)$. Note first that for all n, $f(y_n) \leq \bar{f}$, since $\bar{f}$ is an upper bound for the set $\{f(x) : x \in X\}$. Moreover, it follows from the construction of $\{x_n\}$ that for all n, $f(y_n) = f(x_{\tau(n)}) > \bar{f} - 1/\tau(n) \geq \bar{f} - 1/n$ (by the Fact we proved at the beginning of the lecture, i.e., $\tau(n) > n$, for all n). It follows therefore that for an arbitrarily chosen $\epsilon > 0$, we can pick $N_\epsilon \in \mathbb{N}$, $N_\epsilon > 1/\epsilon$, so that $1/\tau(N_\epsilon) \leq 1/N_\epsilon < \epsilon$. To complete the proof, observe that for all $n > N_\epsilon$,

$$f(y_n) = f(x_{\tau(n)}) \in (\bar{f} - 1/\tau(n), \bar{f}] \subset (\bar{f} - 1/\tau(N_\epsilon), \bar{f}] \subset (\bar{f} - \epsilon, \bar{f}] \subset (\bar{f} - \epsilon, \bar{f} + \epsilon) = B(\bar{f}, \epsilon)$$

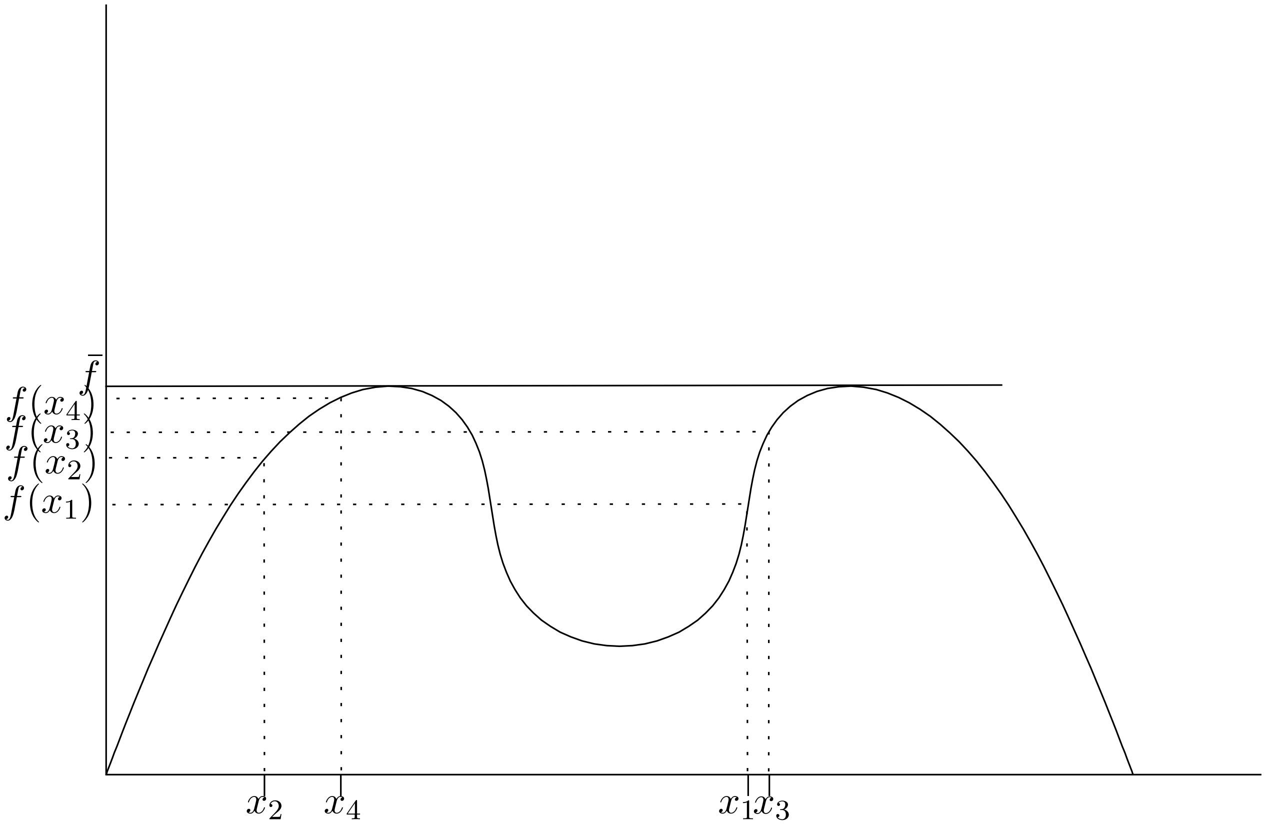


FIGURE 1. Why (x_n) isn't enough: you need to pick a subsequence (y_n)

So we have, $(f(y_n)) \to f(y) \in \mathbb{R}$ and $(f(y_n)) \to \bar{f} \in \mathbb{R}$. Since a sequence can have only one limit,[1] $f(y) = \bar{f}$. Since $\bar{f}$ is an upper bound for $\{f(x) : x \in X\}$, we now have $f(y) \geq f(x)$, for all $x \in X$. □

A common source of puzzlement is: since $f(x_n)$ already converges to $\bar{f}$, why do I need to pick a subsequence (y_n) and show that $f(y_n)$ also converges to $\bar{f}$? The reason is that (x_n) doesn't necessarily converge to some $x \in X$, so I can't invoke continuity to establish that $f(x_n)$ converges to $f(x)$, for some $x \in X$. Here are two examples that illustrate conclusively (I hope) why you absolutely have to pick the subsequence.

(1) Consider the function f and sequence (x_n) graphed in Fig. 1. The example illustrates

[1] We have noted this before but haven't proved it. It's a good exercise to prove it.

the point that even though the sequence $\big(f(x_n)\big)$ converges to $\bar{f}$, the sequence (x_n) doesn't converge. However, (x_n) has two convergent subsequences, each of which work fine.

(2) let $X = (0,1)$ and consider $f : X \to \mathbb{R}$, defined by $f(x) = x$. Clearly f doesn't attain a maximum on X. The theorem doesn't apply because $X = (0,1)$ isn't compact. Here's where the proof would break down if we tried to apply it. Using the notation of the proof, $\bar{f} = 1$. Pick $x_n = 1 - 1/(n+1)$ and note that for all n, $f(x_n) > \bar{f} - 1/n$. But we can't go past this point, because without compactness, e can't pick a subsequence (y_n) and $y \in X$ such that $y_n \to y$. The point of the example is that having the sequence x_n such that the $f(x_n)$'s approach $\bar{f}$ doesn't do us much good, without further help.

1.10. **An open set formulation of continuity**

In analytic treatments of continuity topics, the concepts are typically introduced with a "sequential formulation,"—i.e., someting like, this sequence converges implies that sequence converges—before turning to other, slightly more abstract formulations, typically involving open sets, sometimes called "open set formulations." Our next result establishes for continuity the relationship between the sequential formulation (used above) and the open set formulation.

Definition: Given a mapping $f : X \to Y$, and $O \subset Y$, $f^{-1}(O)$ is the subset of X that f maps into O, i.e., $f^{-1}(O) = \{x \in X : f(x) \in O\}$. $f^{-1}(O)$ is called the inverse image of O under f.

The result is that a function is continuous iff the inverse image of every open subset of the range of the function is an open set in the domain.

Theorem: A function $f : X \to Y$ is continuous iff for every open set $O \subset Y$, $f^{-1}(O)$ is an open subset of X.[2]

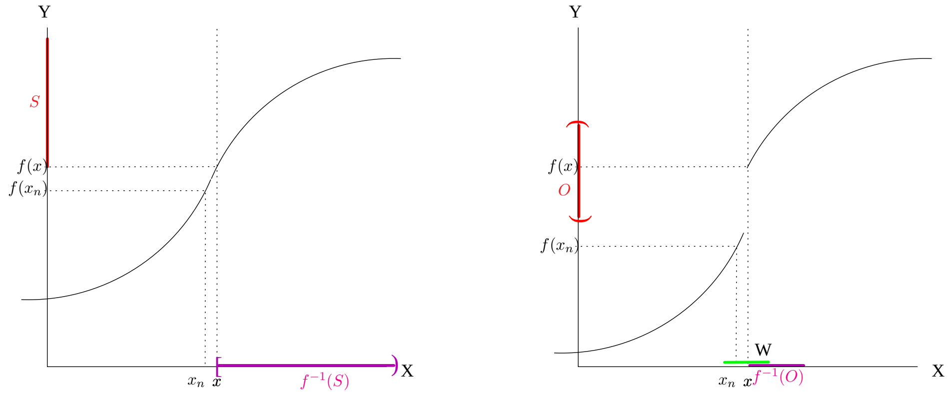


FIGURE 2. f is continuous iff S open $\Longrightarrow f^{-1}(S)$ open

Intuition for the proof: See Fig. 2. In the left panel of the figure, we have a continuuous function; in the right, the function is discontinuous. Consider the left panel and pick a set S in the range such that its inverse image $f^{-1}(S)$ is *not* open. We first prove the "if" part of the theorem, by showing that if f is continuous, then S can't be open. The right hand panel illustrates the "only if" part: we'll show that if f is *not* continuous, we can find an open set in the range whose inverse image (i.e., corresponding set in the domain) is a non-open set. (NOTE: in the left panel, we choose a *non-open* set in the *domain*; in the right panel, we choose an *open* set in the *range*.)

- Since $f^{-1}(S)$ is not open, it contains a boundary point. Call this boundary point x.
- If x is a boundary point, it means that there are points arbitrarily close to it that *don't* get mapped into S, e.g., x_n. Indeed we can pick (x_n) such that $x_n \to x$, and none of the x_n's map into S, i.e., $f(x_n) \notin S$, for all n.
- Since f is continuous, $f(x_n) \to f(x)$.

[2] It is useful sometimes to formulate theorems in the formal language of logic. This theorem is a good example. First I define three "statements"
(1) given a function f, the statement $C(f)$ is true if f is continuous
(2) given a set O, the statement $A(O)$ is true if O is open
(3) given a function-set pair (f, O), the statement $B(f, O)$ is true if $f^{-1}(O)$ is open
Our theorem can now be written as

$$\forall f, \Big(C(f) \Leftrightarrow \big(\forall O, A(O) \Rightarrow B(f, O)\big)\Big)$$

We'll now make two different kinds of contra-positive arguments.
(1) To prove the $\Rightarrow$ part the theorem, we'll show: $\forall f$, given $C(f)$, $\forall O, \neg B(f, O) \Rightarrow \neg A(O)$.
(2) To prove that $\Leftarrow$ part the theorem, we'll show: $\forall f$, given $\neg C(f)$, $\exists O$ s.t. $(A(O) \wedge \neg B(f, O))$
This is a great exercise for getting the hang of formal logical relationships.

- Hence $f(s)$ is a boundary point of S.
- Hence S is boundary point of S and S is not open.

Now let f be discontinuous at S. As you can see from the figure, we can how move S down a bit into the "hole" that's created by the discontinuity. Now chop off the bottom edge of S to create the open set O, and it *still* maps into the same set as S in the domain.

We'll now make this precise.

Proof:

(1) to prove the "if" part of the theorem, we need to show that if f is continuous, then the inverse image of any open sets is open. Fix an arbitrary set $S \subset Y$ such that $f^{-1}(S)$ isn't open in X. We'll argue that if f is continuous, then S isn't open in Y. This will prove that when f is continuous, S open in Y implies $f^{-1}(S)$ is open in X.
   - if $f^{-1}(S)$ isn't open there must exist a point x in $f^{-1}(S)$ (i.e. such that $f(x) \in S$) which is a boundary point of $f^{-1}(S)$.
   - i.e., there's a sequence of points x^n converging to x all of which get mapped to points outside of S, that is, for all n, $f(x^n) \notin S$.
   - since f is continuous, $f(x^n)$ must converge to $f(x)$.
   - but this means that $f(x)$ is a boundary point of S.
   - conclude that S isn't open

(2) We'll now prove the "only if" part of the theorem, for the case $Y = \mathbb{R}$. We need to show that if the inverse image of each open set is open, then f is continuous. We'll show that if f is not continuous, then there exists an open set $O \subset Y$ such that $f^{-1}(O)$ isn't open in X.
   - if f isn't continuous, there exists $x \in X$, a sequence $\{x_n\}$ in X, $\epsilon > 0$ and, for all $n \in \mathbb{N}$, an integer $k_n > n$ such that $|f(x_{k_n}) - f(x)| > \epsilon$. Let $O = (f(x) - \epsilon, f(x) + \epsilon)$.

Clearly $f(x) \in O$ so that $x \in f^{-1}(O)$. We'll show that x is not an interior point of $f^{-1}(O)$ and conclude that $f^{-1}(O)$ is not open.

- pick an arbitrary open set W containing x. Since $\{x_{k_n}\}$ converges to x, there exist n sufficiently large that $x_{k_n} \subset W$. But since by assumption, $f(x_{k_n}) \notin O$, it follows that $x_{k_n} \notin f^{-1}(O)$. Since W was chosen arbitrarily, we have established that there does *not* exist an open set which contains x and is itself contained in $f^{-1}(O)$. Conclude that $f^{-1}(O)$ is not open in X.

1.11. **Upper and Lower Hemi continuous correpondences**

Consider a correspondence ξ mapping a metric space S to a metric space T. (The standard notation denoting a correspondence is $\xi : S \twoheadrightarrow T$. The $\twoheadrightarrow$ (as opposed to $\rightarrow$) signifies that it is not necessarily the case that for every $s \in S$, $\{\xi(x)\}$ is a singleton subset of T. $\{\xi(x)\}$ may be any set at all, including the empty set.) For correspondences, we need to generalize the notion of an inverse image.

Definition: Given a set $O \subset T$, the *upper inverse image* of O, denoted $\bar{\xi}^{-1}(O)$, is the set $\{s \in S : \xi(s) \subset O\}$. This is also known as the *strong* inverse image.

Definition: Given a set $O \subset T$, the *lower inverse image* of O, denoted $\underline{\xi}^{-1}(O)$, is the set $\{s \in S : \xi(s) \cap O \neq \emptyset\}$. This is also known as the *weak* inverse image.

Note that the requirement of a non-empty intersection with O is much weaker than the requirement of containment in O, so that for every subset O of the range of the correspondence, the upper inverse is contained in the lower inverse. Fig. 3 illustrates the two concepts:

(1) the red set is the graph of the correspondence ξ.

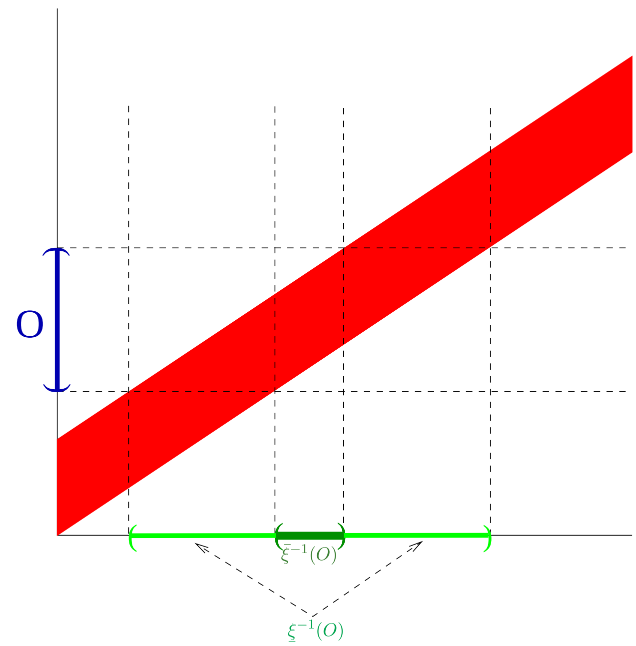


FIGURE 3. Upper and Lower inverse images of ξ

(2) now consider the (open) set in the range, denoted by the blue interval O.

(3) the *upper inverse image* of O, $\bar{\xi}^{-1}(O)$, is the small dark green interval in the domain. Verify that for this example it happens to be an *open* interval.

(4) the *lower inverse image* of O is the larger light green interval in the domain. Verify that for this example it is also an open interval.

We can now define upper- and lower-hemi-continuity.

Definition: ξ is said to upper hemi continuous if for every open set $O \subset T$, the upper inverse image of O, $\bar{\xi}^{-1}(O) \subset S$, is an open set.

Definition: ξ is said to be lower hemi continuous if for every open set $O \subset T$, the lower inverse image $\underline{\xi}^{-1}(O) \subset S$, is an open set.

Definition: ξ is said to be continuous if it is both upper- and lower-hemi-continuous.

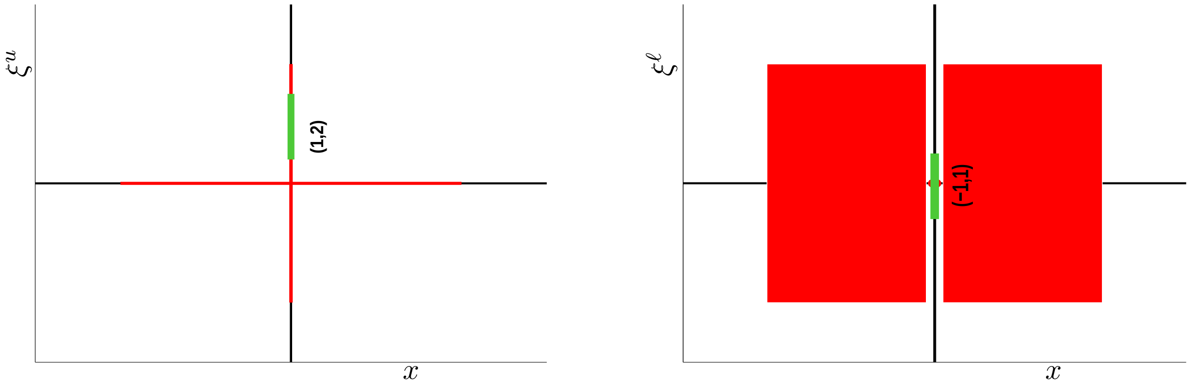


FIGURE 4. ξ^u and ξ^ℓ

To see the difference between the two definitions, compare the two "mirror image" correspondences, $\xi^u : \mathbb{R} \twoheadrightarrow \mathbb{R}$ and $\xi^\ell : \mathbb{R} \twoheadrightarrow \mathbb{R}$, defined by

$$\xi^u = \begin{cases} \mathbb{R} & \text{if } s = 0 \\ \{0\} & \text{if } s \neq 0 \end{cases} \qquad \xi^\ell = \begin{cases} \{0\} & \text{if } s = 0 \\ \mathbb{R} & \text{if } s \neq 0 \end{cases}$$

the graph of ξ^u has a monstrous "jump" at zero in the domain, and the graph of ξ^ℓ has a monstrous "hole" at zero in the domain. More precisely,

(1) consider the open interval $(1, 2)$, and observe the lower inverse image, $\underline{\xi}^u(1, 2)$, is the *closed* set $\{0\}$, establishing that ξ^u is not lhc;
(2) consider the open interval $(-1, 1)$, and observe that the upper inverse image, $\bar{\xi}^\ell(-1, 1)$, is the *closed* set $\{0\}$, establishing that ξ^ℓ is not uhc.

1.11.1. *Concepts related to upper hemicontinuity.*

Theorem: ξ is upper hemi continuous iff for every $\bar{s} \in S$ and every neighborhood U of $\xi(\bar{s})$ there is a neighborhood V of $\bar{s}$ such that $\xi(s) \subset U$ for every $s \in V$.

Proof: Excellent exercise to see if you can extend the logic of the continuity proof on p. 8. Sketch of the idea:

(1) "if" or "$\Longleftarrow$" (nbd condition above implies the relevant property of upper inverse image):

- suppose that $\bar{\xi}^{-1}(O)$ is *not* an open set, for some open set $O \subset T$.
- this means that $\bar{\xi}^{-1}(O)$ contains a boundary point, call it $\bar{s}$.
- i.e., for any nbd V of $\bar{s}$, there exists $s \in V$ such that $\xi(s) \not\subset O$. Now since O is open, it is a neighborhood of itself.
- thus, we've established that there *does not* exist a neighborhood V of $\bar{s}$ such that $\xi(s) \subset U$ for every $s \in V$.

(2) "only if" or "$\Longrightarrow$" (relevant property of upper inverse image implies above nbd condition): Do it as an exercise.

To illustrate, look again at Fig. 4.

(1) in the left panel, there are cases to consider
  (a) $\bar{s} = 0$: in this case the only nbd U of $\xi(\bar{s})$ is $\mathbb{R}$, so this is easy
  (b) $\bar{s} \neq 0$: in this case, $\xi(\cdot)$ is constant on some neighborhood V of $\bar{s}$, so this is again easy.

(2) in the right panel, let $\bar{s} = 0$ and let U be the green nbd, $(-1, 1)$: clearly, there is no nbd V of 0 such that for all $s \in V$, $\xi(s) \in U$.

The next theorem gives an equivalent sequential formulation of the concept:

Theorem: The correspondence ξ is u.h.c iff for every $\bar{s} \in S$, every sequence (s_n) converging to $\bar{s}$ and every sequence (t_n) with $t_n \in \xi(s_n)$, there is a convergent subsequence of (t_n) such that $\lim_n t_n \in \xi(\bar{s})$.

Definition: A correspondence $\xi : S \twoheadrightarrow T$ is said to be *closed-valued* if for every $s \in$S, $\xi(s)$ is a closed set.

Similarly, we define a correspondence to be *compact-valued* if for every $s \in$S, $\xi(s)$ is a compact set.

Given $\xi : S \twoheadrightarrow T$, the *graph* of ξ is defined as $\{(s,t) \in S \times T : t = \xi(s)\}$. This is a set, like any other set, and it may be open, closed or neither.

Definition: A correspondence $\xi : S \twoheadrightarrow T$ is said to have a *closed graph* if its graph is closed (duh).

Note the relationship between the concepts of being closed-valued and having a closed graph: closed graph correspondences are necessarily closed-valued but the converse is not necessarily true.

The concept of upper-hemi-continuity is similar to, but stronger than, the property of having a closed graph. We have the following relationship.

Theorem: If T is compact, then a closed-valued correspondence $\xi : S \twoheadrightarrow T$ is u.h.c iff its graph is closed.

To see the role of compactness of T, consider the correspondence $\xi : \mathbb{R} \twoheadrightarrow \mathbb{R}$ defined by

$$\xi(s) = \begin{cases} \{0\} & \text{if } s = 0 \\ \{0, 1/s\} & \text{if } s \neq 0 \end{cases} \tag{2}$$

Look at the graph of this function: clearly it's closed in $\mathbb{R}^2$, i.e., it contains all its boundary points. (Indeed, in this example, *every* element of the graph is a boundary point.) But it fails the definition of u.h.c, i.e., the neighborhood $U = (-1, 1)$ of $\xi(0) = \{0\}$ has the property that there is no neighborhood V of $s = 0$ such that $\xi(V) \subset U$. (Note that the condition we are imposing here (i.e., T is compact) is much stronger than the condition that ξ is *compact-valued*. The latter property requires only that for every $s \in S$, $\xi(s)$ is a compact set.)

Why does the relationship between closed-graphness and u.h.c. break down in the correspondence depicted in (2)? It's the same idea that we've seen many times: (e.g., $\mathbb{R}$ is closed since it contains all of its accumulation points. Infinity isn't an accumulation point, so we don't have to worry that it isn't "contained" in $\mathbb{R}$.) Similarly, in the case of ξ, sequences in the graph go off into outer space, but these sequences don't accumulate to anything, so we don't have to worry about them. *But* when the graph is contained in a compact set, every sequence *has* to accumulate to something.

So the closed-graph property does not imply u.h.c. Nor is it the case that u.h.c. implies a closed graph. To see this, consider the correspondence $\xi : \mathbb{R} \twoheadrightarrow \mathbb{R}$, defined by, for $s \in \mathbb{R}$, $\xi(s) = (0, 1)$. This function is u.h.c. but it is does not have a closed graph.

1.11.2. *An alternative definition of lower hemicontinuity.*

As usual, we'll state the alternative definition as a theorem.

Theorem: ξ is lower hemi continuous iff for every $\bar{s} \in S$, and every open set $G \subset T$ with $G \cap \xi(\bar{s}) \neq \emptyset$, there exists a neighborhood Z of $\bar{s}$ such that $G \cap \xi(z) \neq \emptyset$, for every $z \in Z$.

Once again, look at Fig. 4:

(1) in the left panel, let G be the green set on the vertical axis and let Z be any nbd of zero. Clearly, $G \cap \xi(z) = \emptyset$, for every $z \in Z$, establishing that ξ^u is not l.h.c.
(2) in the right panel, there are two cases to consider.
    (a) $\bar{s} = 0$: let G denote the green set (-1,1); Clearly, for any nbd Z of zero, $G \cap \xi(z) \neq \emptyset$ for any $z \in Z$.
    (b) $\bar{s} \neq 0$: in this case, $\xi(\cdot)$ is constant on some neighborhood V of $\bar{s}$, so this is again easy.

And the sequential formalation:

Theorem: The correspondence ξ is l.h.c iff for every $\bar{s} \in S$, any $t \in \xi(\bar{s})$, and any sequence (s_n) converging to $\bar{s}$, there exists a sequence (t_n) such that $t_n \in \xi(s_n)$ and $\lim_n t_n = t$.

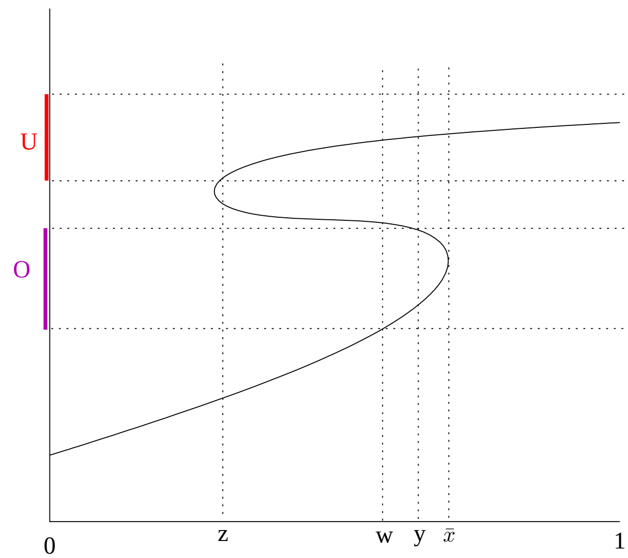


FIGURE 5. $\xi : [0,1] \twoheadrightarrow \mathbb{R}$

1.11.3. *Some examples.* Question: We've noted that unlike a function, a correspondence may be empty valued at some point in its domain. So consider the correspondence $\xi : \mathbb{R} \twoheadrightarrow \mathbb{R}$ defined by

$$\xi(s) = \begin{cases} \emptyset & \text{if } s = 0 \\ \{1/s\} & \text{if } s \neq 0 \end{cases}$$

Is this upper-hemi-continuous? Is it lower-hemi-continuous?

Question: More generally, under what conditions can a upper-hemi continuous correspondence $\xi : S \twoheadrightarrow \mathbb{R}$ satisfy $\xi(s) = \emptyset$, for some $s \in S$? What about a lower-hemi-continuous correspondence?

To get an idea of the answer, let $S' = \{s \in S : \xi(s) \neq \emptyset\}$ and let $\bar{s}$ is a boundary point of S'. If ξ is upper-hemi-continuous, what must be true about $\xi(\bar{s})$? What if ξ is lower-hemi-continuous? In this case, the key point to note is that if $\xi(\bar{s}) = \emptyset$, then there exists no open set $G \subset T$ with the property that $G \cap \xi(\bar{s}) \neq \emptyset$.

Question: Consider the correspondence $\xi : [0,1] \twoheadrightarrow \mathbb{R}$ whose graph is drawn in Fig. 5 below. What can we say about its upper- and lower-hemi-continuousness?

(1) first look at its graph. What can we say about it? What does this imply?
(2) now look at the set U in the range of the correspondence:
- what is its upper inverse image? $\bar{\xi}^{-1}(U) = (\bar{x}, 1]$. Is this set open?

- what about its lower inverse image? $\underline{\xi}^{-1}(U)$ includes points to the left of $\bar{x}$, i.e., $(z, \bar{x})$ as well as points to the right of $\bar{x}$. That is, $\underline{\xi}^{-1}(U) = (z, 1]$.

(3) now consider the inverse images of $U \cup O$

- what is its upper inverse image? $\bar{\xi}^{-1}(U \cup O) = (y, 1]$.
- what is its lower inverse image? $\underline{\xi}^{-1}(U \cup O) = (z, 1]$

(4) finally consider the inverse images of O

- what is its upper inverse image? $\bar{\xi}^{-1}(O)$ is empty.
- what is its lower inverse image? $\underline{\xi}^{-1}(O)$ is the *not*-open interval as $(w, \bar{x}]$.

From the argument so far (from 1 and 4 respectively), we know that ξ is lower-hemi-continuous, but it looks like it is upper-hemi-continuous. (We certainly haven't *proved* the latter.)

1.12. **Summary: the three formulations (courtesy of Steve Buck)**

Remember: any one of the three formulations of each concept can be used as the definition of that concept However, once you pick one of them as your definition then the other two statements follow as "if and only if" theorems.

1(a) Sequential formulation of continuity. A function $f : X \to \mathbb{R}^n$ is called *continuous* at $x_0 \in X$ if whenever $\{x_m\}_{m=1}^{\infty}$ converges x_0 then $\{f(x_m)\}$ converges to $f(x_0)$; the function f is continuous if it is continuous at x for every $x \in X$.

1(b) Neighborhood formulation of continuity. A function $f : X \to Y$ is called *continuous* if for every $\overline{x} \in X$ and every neighborhood U of $f(\overline{x})$ there is a neighborhood V of $\bar{x}$ such that $f(x) \in U$ for every $x \in V$.

1(c) Inverse image formulation of continuity. A function $f : X \to Y$ is called *continuous* if for every open set $O \subset Y$, $f^{-1}(O)$ is an open set of X.

2(a) Sequential formulation of upper hemicontinuity. A correspondence $\xi : S \twoheadrightarrow T$ is called *upper hemicontinuous* if for every $\overline{s} \in S$, every sequence $\{s_n\}$ converging to $\overline{s}$ and every sequence $\{t_n\}$ with $t_n \in \xi(s_n)$, there is a convergent subsequence of $\{t_n\}$ such that the $\lim_{n\to\infty} t_n = \overline{t} \in \xi(\overline{s})$.

2(b) Neighborhood formulation of upper hemicontinuity. A correspondence $\xi : S \twoheadrightarrow T$ is called *upper hemicontinuous* if for every $\overline{s} \in S$ and every neighborhood U of $\xi(\overline{s})$ there is a neighborhood V of $\overline{s}$ such that $\xi(s) \subset U$ for every $s \in V$.

2(c) Inverse image formulation of upper hemicontinuity. A correspondence $\xi : S \twoheadrightarrow T$ is called *upper hemicontinuous* if for every open set $O \subset T$, the upper inverse image of O, $\overline{\xi}^{-1}(O) \subset S$, is an open set.

3(a) Sequential formulation of lower hemicontinuity. A correspondence $\xi : S \twoheadrightarrow T$ is called *lower hemicontinuous* if for every $\overline{s} \in S$, any $t \in \xi(\overline{s})$, and any sequence $\{s_n\}$ converging to $\overline{s}$, there exists a sequence $\{t_n\}$ such that $t_n \in \xi(s_n)$ and $\lim_{n\to\infty} t_n = t$.

3(b) Neighborhood formulation of lower hemicontinuity. A correspondence $\xi : S \twoheadrightarrow T$ is called *lower hemicontinuous* if for every $\overline{s} \in S$, and every open set $G \subset T$ with $G \bigcap \xi(\overline{s}) \neq \emptyset$, there exists a neighborhood Z of $\overline{s}$ such that $G \bigcap \xi(z) \neq \emptyset$, for every $z \in Z$.

3(c) Inverse image formulation of lower continuity. A correspondence $\xi : S \twoheadrightarrow T$ is called *lower hemicontinuous* if for every open set $O \subset T$, the lower inverse image of O, $\underline{\xi}^{-1}(O) \subset S$, is an open set.

1.13. **Application: Berge's Theorem and the demand correspondence**

For economists, by far the most familiar application of correspondences is Berge's maximum theorem, which is used to established that demand is an upper-hemi-continuous correspondence. Below is a version of the theorem, though certainly not the most general version: For concreteness, think of $\beta(\cdot)$ as a budget set, $\xi(\cdot)$ as a demand correspondence, $u(\cdot)$ as a utility function and $V(\cdot)$ as the indirect utility function corresponding to $u(\cdot)$, i.e., mapping each price vector to the maximized value of $u(\cdot)$ on the budget set defined by $\mathbf{p}$.

Theorem: Given $P \subset \mathbb{R}^n$ and $X \subset \mathbb{R}^n$, let $\beta : P \twoheadrightarrow X$ be a continuous, compact-valued correspondence. Let $u : X \to \mathbb{R}$ be a continuous function. Define the "argmax" correspondence $\xi : P \twoheadrightarrow X$ by

$$\xi(\mathbf{p}) \quad = \quad \{\mathbf{x} \in \beta(\mathbf{p}) : \mathbf{x} \text{ maximizes } u(\cdot) \text{ on } \beta(\mathbf{p})\}$$

and the "value" function $V : P \to \mathbb{R}$ by

$$V(\mathbf{p}) \quad = \quad u(\mathbf{x}) \text{ for any } \mathbf{x} \in \xi(\mathbf{p})$$

Then ξ is u.h.c. and compact-valued, while V is continuous.

The following example, depicted in Fig. 6, illustrates why the result is needed and how it works. The domain of the budget and domain correspondences is the interval at the top of the figure: the space $P \subset \mathbb{R}_{++}$ denotes the price of good 2 *relative to* good 1. (It's important that the bottom end of the interval P is a positive number. Why??) The consumer has an endowment $\boldsymbol{\omega} \in \mathbb{R}^2_{++}$. The budget set is $\beta(\mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^2_+ : \mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \boldsymbol{\omega}\}$. Three budget sets for relative prices 0.5, 1 and 2 are drawn at the bottom of the figure. In each of the three bottom panels, the two parallel (red) lines denote the consumer's indifference curves, while the asterisk denotes the endowment

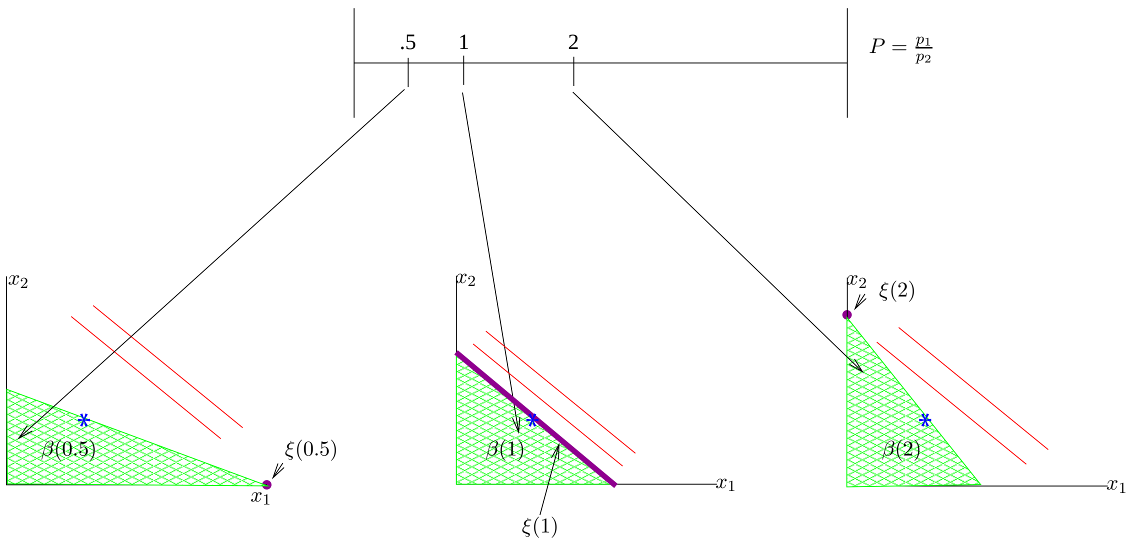


FIGURE 6. The demand correspondence is u.h.c

vector $\boldsymbol{\omega}$ (which is necessarily always in the budget set). Except when $\frac{p_1}{p_2}$ is unity, the budget correspondence is single valued, but it "jumps" from the northwest corner of the budget set to the south-east corner as relative prices pass through unity, at which point the consumer is indifferent between all goods on the outer boundary of the budget set, i.e., $\xi(1)$ is multiple valued. Note that indeed ξ is compact valued. You should verify that it is upper- *but not lower-* hemi-continuous. Also, it should be intuitively clear that $V(\cdot)$ is continuous at unity, even though ξ is not.

Now let's consider a more abstract question. Let $P \subset \mathbb{R}^n_{++}$ and $X \subset \mathbb{R}^n_{+}$. Now consider the budget correspondence $\beta : P \times \mathbb{R}_{++} \twoheadrightarrow X$, where for $(\mathbf{p}, y) \in P \times \mathbb{R}_{++}$, $\beta(\mathbf{p}, y) = \{\mathbf{x} \in X : \mathbf{p} \cdot \mathbf{x} \leq y\}$. In order to apply Berge's theorem to this correspondence, we need to show that $\beta(\cdot, \cdot)$ is a continuous corresponence, i.e., it is both lower and upper-hemi-continuous. That is, we need to show that for an arbitrarily chosen open set $O \subset X$,

(1) $\underline{\beta}^{-1}(O)$ is open in $P \times \mathbb{R}_{++}$.
(2) $\bar{\beta}^{-1}(O)$ is open in $P \times \mathbb{R}_{++}$.

Lower-hemi-continuity turns out to be easier to show, so we'll talk about this first.

To establish $\underline{\beta}^{-1}(O)$ is open in $P \times \mathbb{R}_{++}$, we might proceed as follows: pick an arbitrary pair $(\mathbf{p}, y) \in \underline{\beta}^{-1}(O)$, i.e., such that $\beta(\mathbf{p}, y) \cap O \neq \emptyset$, or, in other words, there exists $\mathbf{x} \in O$, $\mathbf{x} \neq 0$, such that $\mathbf{p} \cdot \mathbf{x} \leq y$. (How do we know that such a $\mathbf{x}$ exists?) To prove that $\underline{\beta}^{-1}(O)$ is open, we need to show that $(\mathbf{p}, y)$ is not a boundary point of $\underline{\beta}^{-1}(O)$. (Think what this statement means; it's not entirely transparent. It means that for any pair $(\mathbf{p}', y')$ that is sufficiently close to $(\mathbf{p}, y)$, there will be *some* commodity bundle in O that is also in $\beta(\mathbf{p}', y')$).

Since O is open in X, there exists $\epsilon > 0$ such that $\underline{\mathbf{x}} = (1 - \epsilon)\mathbf{x} \in O$. (Please verify this, noting in particular that the i'th component of $\mathbf{x}$ could be zero, in which case $(1 - \epsilon)\underline{x}_i = x_i = 0$. Note also that $\underline{\mathbf{x}} \neq \mathbf{x}$. Why not?) One way to proceed is to find $\delta > 0$ such that when $(\bar{\mathbf{p}}, \underline{y}) = \big((1 + \delta)\mathbf{p}, (1 - \delta)y\big)$, then $\bar{\mathbf{p}} \cdot \underline{\mathbf{x}} \leq \underline{y}$, meaning that $\underline{\mathbf{x}} \in \beta(\bar{\mathbf{p}})$, or in other words $\beta(\bar{\mathbf{p}}, \underline{y}) \cap O \neq \emptyset$. In words, because $\mathbf{x} \in O$, there's a smaller bundle $\underline{\mathbf{x}}$ that is also in O; now if you lower income and raise all prices by a teeny enough bit, this smaller bundle will still be affordable. (You should check that this argument holds whether or not your starting vector $\mathbf{x}$ has any zero components.) To complete the proof, you would need to go from the fact that $\beta(\bar{\mathbf{p}}, \underline{y}) \cap O \neq \emptyset$ to show that there is an entire open neighborhood U about $(\mathbf{p}, y)$ with the property that $(\mathbf{p}', y') \in U$ implies $\beta(\mathbf{p}', y') \cap O \neq \emptyset$. This step is pretty straightforward

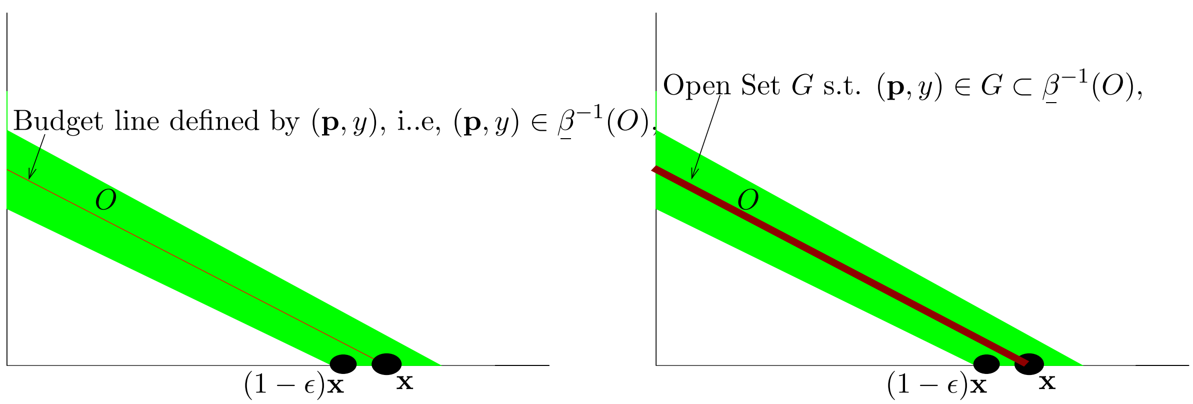
(1) 

FIGURE 7. Proof that the budget correspondence is l.h.c

(2) To show that $\bar{\beta}^{-1}(O)$ is open in $P \times \mathbb{R}_{++}$, we could try to proceed in a parallel fashion: pick an arbitrary pair $(\mathbf{p}, y) \in \bar{\beta}^{-1}(O)$, i.e., such that $\beta(\mathbf{p}, y) \subset O$. To prove that $\bar{\beta}^{-1}(O)$ is open, we need in this case to show that $(\mathbf{p}, y)$ is not a boundary point of $\bar{\beta}^{-1}(O)$. But this step is much harder than step (1), for the following reason: in (1), we started with a *single* $\mathbf{x} \in \beta(\mathbf{p}, y)$ with the property that $\mathbf{x} \in O$ and we needed to show that if $(\mathbf{p}', y')$ were chosen sufficiently close to $(\mathbf{p}, y)$, then there would be a *single* $\mathbf{x}' \in \beta(\mathbf{p}', y')$ such that $\mathbf{x}' \in O$. In step (2), our starting point is that *the entire set* $\beta(\mathbf{p}, y)$ is in O and we need to show that if $(\mathbf{p}', y')$ is sufficiently close to $(\mathbf{p}, y)$, then the *entire set* $\beta(\mathbf{p}', y')$ will again be contained in O. Sounds like a lot of work, indeed I'm getting a headache just thinking about how to do this directly.

So lets try a completely different route to get to the same place, in particular we'll use a contra-positive argument. We'll assume that $\bar{\beta}^{-1}(O)$ is *not* open in $P \times \mathbb{R}_{++}$ and conclude that O is *not* open in X. Specifically, assume that $(\mathbf{p}, y)$ is a boundary point of $\bar{\beta}^{-1}(O)$, i.e., that there exists a sequence $\{(\mathbf{p}^n, y^n)\}$ which converges to $(\mathbf{p}, y)$ and a sequence $\{\mathbf{x}^n\}$ in X with the property that for each n, $\mathbf{x}^n \in \beta(\mathbf{p}^n, y^n)$ but $\mathbf{x}^n \notin O$. Happily for us, $\{\mathbf{x}^n\}$ contains a convergent subsequence $\{\mathbf{x}^{n_k}\}$. (How do we know this??? You can't just assume it, needs a little work.) Let $\mathbf{x}$ be the limit of this subsequence. Now by assumption, for all k, $\mathbf{p}^{n_k} \cdot \mathbf{x}^{n_k} \leq y^{n_k}$ so that $\mathbf{p} \cdot \mathbf{x} \leq y$. (Check this!!) In other words $\mathbf{x} \in \beta(\mathbf{p}, y) \subset O$. But this means that $\mathbf{x}$ is a boundary point of O, meaning that O is not open. We've thus proved that if $\bar{\beta}^{-1}(O)$ has a boundary point then O has a boundary point also.