ARE211, Fall 2009

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CONTENTS

1. Analysis (cont)

1.8. Topology of $\mathbb{R}^n$ (cont)

1.8.5. *Closure of a Set.* The analog of the interior of a set is the closure of a set.

Definition: Let $A \subset X$. The *closure of* A in X, denoted $\text{cl}(A)$ or $\bar{A}$ in X is the intersection of all closed sets containing A.

Theorem: For $A \subset X$, A is closed in X iff $A = \text{cl}(A)$ in X.

1.8.6. *Boundary of a Set.* A point $\mathbf{x}$ is a boundary point of a set $A \subset X$ if there are points arbitrarily close to $\mathbf{x}$ that are in A *and* if there are points arbitrarily close to $\mathbf{x}$ that are in X but *not* in A.

Example: The set $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. Its boundary is the circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

That is, the boundary is the border between A and $X \backslash A$.

Definition: The set of boundary points of A in X, denoted $\mathrm{bd}(A)$, is the set $\mathrm{cl}(A) \cap \mathrm{cl}(X \backslash A)$.

Theorem: Let $A \subset X$. A point $x \in \mathrm{bd}(A)$ iff $\forall \epsilon > 0$, $\exists y, z \in B(x, \epsilon)$ such that $y \in A$ and $z \in X \backslash A$.

Example: The set $\{1, 2, 3, 4, 5\}$ has *no* boundary points when viewed as a subset of the integers; on the other hand, when viewed as a subset of $\mathbb{R}$, every element of the set is a boundary point.

Theorem: A set $A \subset X$ is *closed* in X iff A contains all of its boundary points.

Note the difference between a boundary point and an accumulation point.

Take the set $A = \{0\} \subset \mathbb{R}$. 0 is a boundary point of A but not an accumulation point. On the other hand, *every* element of the interval $A = (0, 1) \subset \mathbb{R}$ is an *accumulation point* of A, but A contains *none* of the *boundary* points of A.

Though boundary points and accumulation points are resoundingly different *in general*, there is a close connection between the two concepts:

Theorem: Given a set $A \subset X$, a point $x \in X$ that *does not belong to* A is a boundary point of A in X iff it is an accumulation point of A in X.

1.8.7. *Compact Sets.* Importance of compact sets: continuous functions defined on compact sets always attain their extrema.

Up until now, we've defined general notions of open, closed, boundary, etc. Now the notion we define is very special to Euclidean space. There's a general notion of compactness which I'm not going to teach. We are going to focus on $\mathbb{R}^n$.

In order to define a compact set in $\mathbb{R}^n$, we need to generalize the notion of boundedness that I gave you earlier for $\mathbb{R}$.

Definition: For $\mathbf{x} \in \mathbb{R}^n$, the *norm* of $\mathbf{x}$, written $||\mathbf{x}||$, is the Euclidean distance between $\mathbf{x}$ and zero, i.e., $||\mathbf{x}|| = \sqrt{\sum_{i=1}^{n} x_i^2}$.

Definition: A set $A \subset \mathbb{R}^n$ is *bounded* if there exists a real number $b \in \mathbb{R}$ such that $||\mathbf{x}|| \leq b$, for all $\mathbf{x} \in A$. (Note that this definition of bounded is metric-specific!)

Definition: A set $A \subset \mathbb{R}^n$ is *compact* in the Euclidean metric if it is closed and bounded.

1.9. **The Bolzano-Weierstrass Theorem**

We are going to prove one of the most fundamental theorems of analysis:

Theorem: (informally) Every sequence defined on a compact set contains a convergent subsequence.

REVIEW:

(1) recall that a sequence in S is called a *convergent sequence* if it converges to an element of the set S. E.g., the sequence $(1/n)$ in the set $S = (0,2) \subset \mathbb{R}$ is *not* a convergent subsequence because the point in $\mathbb{R}$ that is the limit of the sequence does not belong to the set S that contains the sequence.

(2) Recall the definition of a subsequence: a subsequence of $\{x_1, x_2, ..., x_n, ...\}$ is a sequence $\{y_1, y_2, ..., y_n, ...\}$ if there exists a *strictly increasing* function $\tau : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, $y_n = x_{\tau(n)}$. That is, you construct a subsequence by throwing out elements of the original sequence, but keeping an infinite number of the original elements *and preserving their order.*

Example: Take the sequence $\{-1, 1, -1, 1, ...\} \subset [-1, 1]$. Set $A_1 = [0, 1]$, $A_2 = [1/2, 1]$, $A_3 = [3/4, 1]$, etc. To construct a convergent subsequent, set $\tau(1) = 2$, $\tau(2) = 4$, $\tau(3) = 6$, etc., and note that for all n, $y_n = x_{\tau(n)} \in A_n$

Intuitively, the theorem we're going to prove is obvious: points in the sequence have to bunch up somewhere. Nowhere for them to escape to. The basic idea of the proof is: because the set is bounded, we can construct a Cauchy subsequence of the original sequence; because the set is closed, the point to which the subsequence is trying to converge actually belongs to the set, and so is indeed a limit of the subsequence.

The proof is going to involve two *inductive constructions*: an inductive construction of a sequence involves:

(1) an *initial* step: define the first element of the sequence
(2) an *inductive* step: assume that the n'th element of the sequence has been defined; now define the $n+1$'th element in terms of the n'th.

For example, a first order difference equation is defined by induction: define x_0; then define $x_{t+1} = ax_t + b$.

Here's the intuition for the proof, for the case of a sequence in $\mathbb{R}$:

(1) Because the set S is compact, it's bounded. Pick $b \in \mathbb{R}$ such that S is contained in the interval $[-b/2, b/2]$. (You'll see why we divide by 2 in a minute.) Now consider any sequence in S.
(2) Divide the interval $[-b/2, b/2]$ into two halves; at least one half contains an infinite number of elements in the sequence (possibly both halves). Pick this half (if there are two such halves, pick either).
(3) The inductive step: take the set you've just picked, split it in half, and pick again, according to the same criterion. Keep going forever; note that you now have an infinite sequence of *nested* sets.
(4) Define a subsequence of the original sequence as follows: the first element of the subsequence is the first element of the original sequence that belongs to the first subdivided set.
(5) The second element is the *first* element in the original sequence *that comes after* the one you've just chosen which lies in the second subdivided set, etc.
(6) Now you have constructed an infinite sequence of points with the property that for every N, the entire tail of the sequence (after discarding the first N points) lies in the N'th subdivision.

(7) Since the subdivided intervals are getting smaller and smaller, the sequence is a Cauchy sequence.

(8) Every Cauchy sequence defined in $\mathbb{R}$ converges to a point in $\mathbb{R}$.

(9) The point in $\mathbb{R}$ to which the sequence converges is either an element of the sequence or an accumulation point of the original compact set. Since the set is compact, hence closed, it contains its accumulation points, hence the sequence we've constructed indeed converges to a point in the set.

Theorem: (Bolzano-Weierstrass) A set $A \subset \mathbb{R}^n$ is compact if and only if every sequence $\{\mathbf{x}_n\}_{n=1}^{\infty}$ in A has a convergent subsequence, i.e., there exists a subsequence $\{\mathbf{y}_n\}_{n=1}^{\infty}$ of $\{\mathbf{x}_n\}_{n=1}^{\infty}$ and a point $\mathbf{y} \in A$ such that $\{\mathbf{y}_n\}$ converges to $\mathbf{y}$.

Proof: We'll do the proof for $A \subset \mathbb{R}$ and only in one direction, i.e., we'll show that A compact implies each sequence in A has a convergent subsequence.

Suppose that A is bounded above by $b/2$ and below by $-b/2$: that is, the interval $[-b/2, b/2]$ contains the entire (infinite) number of points in the sequence. Split the set $[-b/2, b/2]$ into two closed subsets of equal length, i.e., into $[-b/2, 0]$ and $[0, b/2]$ and observe that *at least one* of these subsets contains an infinite number of elements of the sequence $\{x_n\}$. Denote by A_1 whichever subset this is. If both have this property, then pick either. The length of the subset A_1 is equal to $b/2$. Now argue by *induction*. Suppose that we've defined a subset A_n of length $b/2^n$ which contains an infinite number of elements of the sequence $\{x_n\}$. (This statement is true for $n = 1$ i.e., length of A_1 is $b/2 = b/2^1$.) We'll construct a closed subset A_{n+1} of length $b/2^{n+1}$ which contains an infinite number of elements of $\{x_n\}$: simply divide A_n in half as before: at least one of the subsets contain an infinite number of elements. Now we've constructed an infinite sequence of subsets, $\{A_n\}$ with the property that for each n, (a) $A_{n+1} \subset A_n$ (i.e., the A_n's are nested) (b) the length of A_n equals $b/2^n$ (c) A_n contains an infinite number of elements of $\{x_n\}$.

Now we'll define a convergent subsequence $\{y_n\}$. Define the strictly increasing sequence τ as follows: let $\tau(1)$ denote the smallest element of the set $\{k \in \mathbb{N} : x_k \in A_1\}$; now assume that $\tau(n)$ has been defined and define $\tau(n+1)$ to be the smallest element of of the set $\{k \in \{\tau(n) + 1, ...,\} : x_k \in A_{n+1}\}$. Note that by construction, $\tau(\cdot)$ is a strictly increasing function of n.

Observe that we now have constructed a subsequence $\{y_n\}$, i.e., $y_n = x_{\tau(n)}$, for each n, with the property that for each n, $y_n \in A_n$. Moreover, since the A_n's are nested, it follows that for all N, and $n, m > N$, $y_n, y_m \in A_N$. That is, $\{y_n\}$ is a Cauchy sequence. Now, we know that every Cauchy sequence in $\mathbb{R}$ converges to a point in $\mathbb{R}$. i.e., there exists $y \in \mathbb{R}$ such that for all $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that for all $n > N$, $d(y_n, y) < \epsilon$. We need to show that $y \in A$. If $y = y_n$, for some n, we're done, since $y_n \in A_n \subset A$. If not, then y is an accumulation point of the set A, since for all $\epsilon > 0$, there exists n such that $y_n \neq y$ and $y_n \in B(y, \epsilon)$. But since A is a closed set, it contains all of its accumulation points. Hence $y \in A$, and we've proved that the subsequence we've constructed converges to a point $y \in A$. $\square$