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1. ANALYSIS (CONT)

1.8. **Topology of  $\mathbb{R}^n$**

1.8.1. *Open Sets.*

Definition: Given a universe  $X$ , a metric  $d$  and  $x \in X$ , the set  $B_d(x, \epsilon|X) = \{y \in X : d(x, y) < \epsilon\}$  is called the  $\epsilon$ -ball about  $x$ .

Important to note that the definition of openness depends on a *metric* and a *universe*. Technically, we should write  $B_d(x, \epsilon|X)$

Examples:

- Suppose  $X = [0, 1]$  and consider the  $\epsilon$ -ball about 1: it is  $(1 - \epsilon, 1]$ .
- Suppose  $d$  is the discrete metric and  $X = \mathbb{R}$ . For  $\epsilon \leq 1$ , the  $\epsilon$ -ball about  $x \in \mathbb{R}$  is the point itself. For  $\epsilon > 1$ , the  $\epsilon$ -ball about  $x \in \mathbb{R}$  is  $\mathbb{R}$ .

More examples:

- Suppose  $X = \mathbb{R}^n$  and  $d$  is the max metric, i.e.,  $d(\mathbf{x}, \mathbf{y}) = \max\{|x_i - y_i| : i = 1, \dots, n\}$ . What's the shape of the ball? It's an  $n$ 'dimensional cube.
- Suppose  $X = \mathbb{R}^n$  and  $d$  is the absolute value metric, i.e.,  $d(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n |x_i - y_i|$ . What's the shape of the ball? It's the  $n$ -dimensional version of a diamond.

From now on, we will usually take for granted that the universe is  $\mathbb{R}^n$  and the metric is the Euclidean metric. Unless we specify that these are otherwise, we will simply write  $B(x, \epsilon)$ .

**Definition:** A set  $A \subset X$  is said to be *open* in  $X$  w.r.t. a metric  $d$  if for every  $x \in A$ , there exists  $\epsilon > 0$  such that  $B_d(x, \epsilon|X) \subset A$ .

**Example:**

- the interval  $(a, b)$  is open in  $\mathbb{R}$  but not open in  $\mathbb{R}^2$ , e.g., when viewed as a horizontal or vertical line in  $\mathbb{R}^2$ .
- the set  $\{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| < 1\}$  is an open set
- the half-open interval  $\{x \in [0, 1] : 1/2 < x \leq 1\}$  is open in  $[0, 1]$  but not open in  $\mathbb{R}$ .
- the singleton set  $\{x\} \subset \mathbb{R}$  is open in  $\mathbb{R}$  w.r.t. the discrete metric.
- the empty set is always open, no matter what: look at the defn: every point in the set must be enclosed by an open ball that's in the set. This defn is certainly satisfied by the empty set!! (As we'll see later, the empty set is also closed.)

**Theorem:** For every  $\epsilon > 0$  and  $\mathbf{x} \in \mathbb{R}^n$ , the  $\epsilon$ -ball about  $x$ ,  $B(\mathbf{x}, \epsilon)$ , is open in  $\mathbb{R}^n$ .

**Proof:** Pretty obvious.

The following theorem contains the most important basic properties of open sets

**Theorem:** The intersection of a finite number of open subsets of  $\mathbb{R}^n$  is an open subset of  $\mathbb{R}^n$ . The union of an *any* collection (i.e., even an infinite number) of open subsets of  $\mathbb{R}^n$  is an open subset of  $\mathbb{R}^n$ .

Note that the intersection of a infinite number of open subsets need not be open:

**Example:**  $X^n = (-1/n, 1/n) \in \mathbb{R}$ . The intersection of  $X^n$ 's, for all  $n$ , is the singleton set  $\{0\}$ , which is not open in  $\mathbb{R}$  in the usual metric.

Suprising things happen with suprising metrics: *every* subset of  $\mathbb{R}$  is an open set w.r.t. the discrete metric: reason is that each scalar in  $\mathbb{R}$  is open in this metric, and any set  $X \subset \mathbb{R}$  is a union of scalars in  $\mathbb{R}$ .

Context is particularly important to us as economists because we are often dealing with functions defined on a restricted domain, and we'll be interested in whether or not a subset of that domain is open or not *in that domain*. For example, often the domain of our functions is  $\mathbb{R}_+$ : subsets of the form  $[0, \epsilon)$  are open in  $\mathbb{R}_+$  but not in  $\mathbb{R}$ .

### 1.8.2. Interior of a Set.

**Definition:** For any set  $A \subset X$  a point  $x \in A$  is called an *interior point* of  $A$  if there is an open set  $U \subset X$  such that  $x \in U \subset A$ . The interior of  $A$  is the collection of all interior points of  $A$  and is denoted  $\text{int}(A)$ . (Note that  $\text{int}(A)$  may be empty.)

As always whether a point is an interior point or not depends on *universe* and *metric*.

- Let  $A = [0, 1] \subset \mathbb{R}$ . In this case,  $\text{int}(A) = (0, 1)$ .
- Let  $A = [0, 1] \subset [0, 1]$ . In this case,  $\text{int}(A) = [0, 1]$ .
- Generally, given any universe  $X$ , the set  $X \subset X$  is equal to  $\text{int}(X)$ .

**Theorem:** For any set  $A \in X$ , the interior of  $A$  w.r.t.  $X$  is the union of all subsets of  $A$  that are open in  $X$ .

**Proof:** Consider a point  $x$  contained in some open subset of  $A$ . Then it's an interior point and hence, immediately, belongs to interior  $A$ . Now consider  $x \in A$  such that  $x$  is not contained in *any* open subset of  $A$ . Then it fails the defn of an interior point.

**Theorem:** A set  $A$  is open in  $X$  iff  $A = \text{int}(A)$ .

1.8.3. *Closed Sets.* Informally, a set is closed if it contains its edges (boundary points).

**Definition:** A set  $B \subset X$  is *closed* in  $X$  if its complement in  $X$  (i.e., the set  $X \setminus B$ ) is open.

Note that  $X$  is closed in  $X$ , since its complement in  $X$ , i.e., the empty set is open.

Sets can be neither open nor closed. An example is the set  $[0, 1] \in \mathbb{R}$ .

1.8.4. *Accumulation Point.* **Definition:** A point  $x \in X$  is called an *accumulation point* of a set  $A \subset X$  if for every  $\epsilon > 0$ , the ball  $B(x, \epsilon | X)$  contains a point  $y \in A$ ,  $y \neq x$ .

**Important note:** At this point, some of my defns depart from the defns that appear in important texts, including Simon and Blume. Don't be too concerned about this. Unless you're super obsessive compulsive, just go with my defns. (As in when you are in Hong Kong you drive on the right side of the road.) If you are super obsessive compulsive and are worried about defns in books that differ from mine, come see me and I'll discuss.

That is, a point is an accumulation point of a set if there are points in the set that are arbitrarily near to  $x$ . But an accumulation point of a set needn't belong to the set.

People are always confused about the relationship between an accumulation point and a limit. Limits are things that *sequences* have; accumulation points are things that *sets* have. (Some books don't necessarily make this distinction, however.)

**Example:** The point  $1 \in \mathbb{R}$  is an accumulation point of the set  $(0, 1) \subset \mathbb{R}$  in the usual metric:  $\forall \epsilon > 0$ ,  $1 - \epsilon/2 \in B(1, \epsilon)$  and  $1 - \epsilon/2 \in (0, 1)$ .

**Example:** Note that a set consisting of *isolated* points (e.g., the integers) contains no accumulation points.

**Example:** More interesting example is the rational numbers  $\mathbb{Q}$ . Turns out that every point in  $\mathbb{R}$  is an accumulation point of  $\mathbb{Q}$  in the usual metric, i.e., there is a rational number arbitrarily close to any real number.

Once again, the notion of accumulation depends on the metric you use.

Think about the discrete metric: take the set  $(0, 1)$ ; what are the accumulation points of this set; there aren't any, since any two points that aren't the same are distance 1 away from each other.

As always, the notion of accumulation depends on the context or universe. Suppose that  $X = (0, 1)$ . Consider the set  $[1/2, 1)$ . What are its accumulation points in  $X$ ? Note that 1 isn't in  $X$ , so isn't an accumulation point in this context. This example provides another instance of how the universe matters. In  $X = (0, 1)$ , all of the points in  $[1/2, 1)$  are accumulation points, so this set is closed in  $(0, 1)$ .

There is a very close relationship between the notion of accumulation and the notion of a closed set.

**Theorem:** A set  $A \subset X$  is *closed* in  $X$  iff  $A$  contains all of its accumulation points.

That is, in order for  $A \subset X$  to be closed, if  $\mathbf{x} \in X$  and there are points arbitrarily close to  $\mathbf{x}$  that are in  $A$ , then  $\mathbf{x}$  must be in  $A$  also.

Note that the integers are closed in  $\mathbb{R}$  under the usual metric, since there are no accumulation points to contain.

Think about the discrete metric. What sets are closed in the discrete metric?

**Theorem:** Every set  $A \subset X$  is *closed* in the discrete metric.

Note that every set  $X \subset X$  is closed with respect to itself. Can't be any accumulation points of  $X$  that aren't included in  $X$ . Also, every point in  $X \subset X$  is an interior point, hence  $X$  is open. Note that this implies that the empty set is both closed and open.