ARE211, Fall 2009

Contents

1. Analysis (cont)

1.6. **Two Preliminary Results on least upper bounds**

We need these little results for the big theorem on Cauchy sequences that follows.

Theorem: (Theorem A for future ref): Given two subsets, P and Q, of the real line, if $P \subset Q$ then $\sup(P) \leq \sup(Q)$ and $\inf(P) \geq \inf(Q)$.

This result is completely obvious, once you look at a picture: if P is contained in Q then then every upper bound to Q must also be an upper bound to P and there may be upper bounds to P that aren't upper bounds to Q. Doing the proof is just a matter of saying the obvious in symbols.

We'll just prove the part about sups. The other proof is parallel. But you should work through it as an exercise, without looking at the notes.

Proof: Pick $P, Q \subset \mathbb{R}$ such that $P \subset Q$. There are two cases to consider:

Case A: $\sup(Q) = \infty$; in this case, there's nothing to prove because *any* possible sup for P has to be less than or equal to infinity.

Case B: $\sup(Q) = \bar{b}$. We'll consider $b > \bar{b}$ and prove that b cannot be a least upper bound for P. This will establish that $\sup(P) \leq \sup(Q)$. Well, $\bar{b} \geq q$, for every $q \in Q$. Now pick $p \in P$. Since $p \in Q$, $\bar{b} \geq p$, Hence $\bar{b}$ is an upper bound for P. Since $b > \bar{b}$, b cannot be a *least* upper bound for P. □

Notice that if the statement of Theorem A had omitted any mention of the real line, then the theorem would have been false. Why?[1]

Theorem: (Theorem B for future ref): Given $S \subset \mathbb{R}$, $b \in \mathbb{R}$ is the least upper bound of S iff b is an upper bound for S and if for every $\epsilon > 0$, there exists $s \in S$ such that $b - s < \epsilon$.

Some of you may have seen Theorem B written as a *definition* of least upper bound. It would be a perfectly good definition, since we are just about to show that it is equivalent to the one I gave you. The point is that there are many equivalent definitions for most mathematical concepts. You can *start* with any one of them, then if you decide you want to use another one, you have to prove, using an if and only if proof, that the other one is equivalent.

Proof of Theorem B: Need to prove this in both directions, i.e., (a) if b satisfies the requirements of the theorem then it is a least upper bound; (b) if it *doesn't* satisfy the requirements of the theorem, then it *isn't* a least upper bound.

Proof of (a): Assume that for every $\epsilon > 0$, there exists $s \in S$ such that $b - s < \epsilon$. We need to show that $b \leq b'$ for every upper bound b' for S. We'll do this by showing that if $b' < b$ then b'

[1] Because the sup would not necessarily exist, so the statement of the theorem would be nonsense.

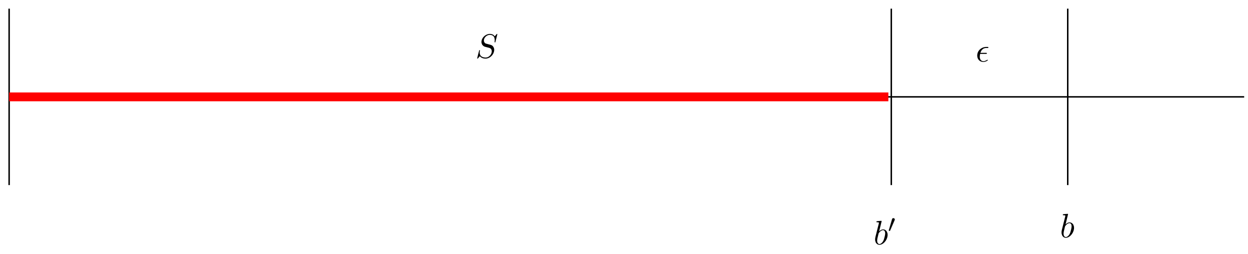

FIGURE 1. Proof of (b)

cannot be an upper bound for S. Choose $\epsilon > 0$ such that $b' = b - \epsilon$. By assumption, $\exists s \in S$ s.t. $b - s < \epsilon = b - b'$, i.e., $-s < -b'$ or $s > b'$ proving that b' isn't an upper bound for S.

Proof of (b): Now assume that one of the requirements of the theorem isn't satisfied. The first requirement is that b is an upper bound for S. If this requirement is violated, then trivally b can't be a least upper bound; Now suppose that the $\forall\epsilon$ condition fails, i.e., $\exists\epsilon > 0$ such that $\forall s \in S$, $b - s \geq \epsilon$ (see Fig. 1 below). We'll show that b cannot be a *least* upper bound for S, i.e., we'll show that there exists $b' < b$ such that b' is an upper bound for S. Pick $b' = b - \epsilon$. By assumption, for all $s \in S$ $b - s \geq \epsilon$, or, equivalently, $b - \epsilon \geq s$. Hence, by definition, $b' = b - \epsilon \geq s$, for all $s \in S$. Hence $b' < b$ is an upper bound for S, establishing that b isn't a *least* upper bound. □

1.7. **Cauchy Sequences**

The notion of a Cauchy sequence is another tool that we will need later.

Definition: A sequence $\{x_n\}$ is called a Cauchy sequence with respect to a metric d if $\forall\epsilon > 0$ $\exists N \in \mathbb{N}$ such that for $n, m > N$, $d(x_n, x_m) < \epsilon$.

That is, the sequence bunches up as n progresses. More specifically, if you have the graph of a Cauchy sequence, then for any ϵ, you can split the sequence up into a head and a tail in which a way that the image of the entire tail of the sequence fits inside an interval of length ϵ,

Example: The sequence $\{x_n\}$ defined by $x_n = (-1)^n/n$.

There is an important difference between a Cauchy sequence and a convergent sequence: a convergent sequence converges to a point; a Cauchy sequence bunches up, but there may be nothing that it converges to. For example, consider the sequence $1 - 1/2, ..., 1 - 1/n, ...$ in the set $(0, 1)$. The sequence satisfies the defn of a Cauchy sequence, but it is not a convergent sequence. The thing it is trying to converge to just isn't in the set.

That is, sequences that are "trying to converge to something" are called Cauchy sequences, whether or not the something they are trying to converge to exists.

Another example of a Cauchy sequence that isn't a convergent sequence is the sequence of *continuous functions* we defined in a previous lecture, $\{f_1, f_2, ...f_n...\}$, where $f_n = \begin{cases} -1 & \text{if } x \leq -1/n \\ nx & \text{if } -1/n < x < 1/n \\ 1 & \text{if } x \geq 1/n \end{cases}$.

While we don't know very much yet about metrics on the space of functions[2], you can see intuitively that the f^n's are *in some sense* getting closer and closer together, i.e., their graphs are becoming indistinguishable from each other. (That is, this sequence is Cauchy with respect to *some* metrics on the space of continuous functions—such as the metric defined in footnote 2—and isn't Cauchy with respect to others.) However, the sequence doesn't have a limit *in the space of continuous functions*: if f were a limit function, it would *have to* have the property that $f(x) = 1$, for all $x > 0$

[2] In class I gave an example of a pseudo-metric on functions, i.e., the absolute value of the difference between the integrals of the functions. Here's an actual metric. given $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$, define $\delta(f, g) = \sup_{x \in X} |(f(x) - g(x))|$. You should check that δ indeed satisfies the four requirements of a metric.

and $f(x) = -1$ for all $x < 0$. But what happens at zero?? Clearly the function can't be continuous. This is a typical example of a sequence of objects that all belong to the same set (i.e., the set of continuous functions), in which the elements get closer and closer together, but don't converge to anything in that original set.

The real number system has the following very special property, that relates Cauchy sequences to convergence. To see how special it is, note that it isn't satisfied either for the rationals $\mathbb{Q}$ or for the space of continuous functions.

Theorem: Every sequence in $\mathbb{R}$ that is Cauchy w.r.t. some metric converges with respect to that metric to a number $x \in \mathbb{R}$.

Note that a sequence in $\mathbb{R}$ may be Cauchy w.r.t. one metric but not with respect to another. As an example, consider the metric ρ (known to generations of ARE students as "Leo's favorite metric") defined on $\mathbb{R}_+ \times \mathbb{R}_+$ by $\rho(x,y) = \begin{cases} 0 & \text{if } x = y = 0 \\ 1/x & \text{if } x \neq y = 0 \\ 1/y & \text{if } y \neq x = 0 \\ |1/x - 1/y| & \text{otherwise} \end{cases}$. Now consider the sequence $\{x_n\}$ defined by $x_n = n$. This sequence is clearly Cauchy[3] w.r.t. ρ—and indeed converges to a point $x \in \mathbb{R}_+$ (which one?)—but, obviously, isn't Cauchy w.r.t. any of the standard metrics on $\mathbb{R}$, e.g., the Euclidean metric.

Why do we care about whether or not a sequence is Cauchy? One reason is that in many cases in economics, we want to prove that some scalar, vector, or function exists. E.g., proving the existence of equilibrium prices for an economy: a price could be a scalar, a vector or in some cases a function. Often, the easiest way to proceed is to construct a Cauchy sequence of, say, equilibrium prices of some sequence of nearby economies, and then argue that that the limit of this sequence is the

[3] Fix $\epsilon > 0$ and let $N > 1/\epsilon$. Now, for $n > m > N$, $\rho(x_n, x_m) = |1/x_n - 1/x_m| = \frac{|x_n - x_m|}{x_m x_n} < \frac{x_n}{x_m x_n} = 1/x_m < 1/N < \epsilon$.

equilibrium we need. But if the limit isn't the same kind of object as the members of the Cauchy sequence, then we can't pursue this route. E.g., our equilibrium price function might be required to be a continuous function of, say, some signal; but the limit of a Cauchy sequence of continuous functions need not be a continuous function, so in this case, the Cauchy approach won't work.

The above theorem is another way of saying that the real number system doesn't have any holes in it. Obvious as it is, the proof involves some work. Essentially the proof formalizes the image I alluded to above: a sequence is Cauchy if we can draw a sequence of lines, getting closer and closer together, just above and just below the graph of the sequence. For any pair of lines, if you start the "tail" of the sequence sufficiently far out along the sequence, the entire tail will fit between the lines. The following argument makes this idea precise.

Proof: We're going to prove the theorem for Euclidean metrics. When defined on $\mathbb{R} \times \mathbb{R}$, *every* Euclidean metric reduces to $d(x, y) = |x - y|$. We first need to show that every Cauchy sequence is bounded below. (It's also bounded above, but we don't need this fact.) Set $\epsilon = 1$; $\exists M \in \mathbb{N}$ such that for $n, m > M$, $d(x_n, x_m) < 1$. The set $\{x_1, ...x_M, x_{M+1}\}$ is a finite set and has a minimum. Call this minimum $\underline{x}$. (Question: why wouldn't the proof below be correct if we kept everything the same, except we defined $\underline{x}$ to be the miminum of the set $\{x_1, ...x_M\}$?) We'll show that $\underline{x} - 1$ is an lower bound for the entire sequence. Obviously, it's a lower bound for the head of the sequence $\{x_1, ...x_M\}$. Now let $m = M + 1$, and note that since $d(x_m, x_n) = |x_n - x_m| < 1$, we have that for all $n > M$, $x_n > x_{M+1} - 1 \geq \underline{x} - 1$. (Note that this is where we use the specific functional form for the metric.) Similarly, the set is bounded above.

Now for each $m \in \mathbb{N}$, let $A_m = \{x_m, x_{m+1}, ...,\}$ and let $a_m = \sup(A_m)$. From Theorem A, the sequence $\{a_m\}_{m=1}^{\infty}$ is a nonincreasing sequence. Moreover the sequence is bounded below (previous paragraph). Hence the sequence $\{a_m\}_{m=1}^{\infty}$ converges to a point $\bar{a} \in \mathbb{R}$. (From the Axiom of Completeness last time.)[4] We'll show $\{x_n\}$ also converges to $\bar{a}$. Fix $\epsilon > 0$.

- pick N_1 sufficiently large that $\forall n, m \geq N_1$, $d(x_n, x_m) < \epsilon/3$ (using the fact that the sequence is Cauchy.)
- pick N_2 sufficiently large that $\forall n \geq N_2$, $d(a_n, \bar{a}) < \epsilon/3$. (using the fact that the sequence of sup's converges to $\bar{a}$).
- let $N = max\{N_1, N_2\}$
- Note that for $n, m > N$ *both* of the following are true:
  - $\forall n, m \geq N$, $d(x_n, x_m) < \epsilon/3$
  - $\forall n \geq N$, $d(a_n, \bar{a}) < \epsilon/3$.

  That is, all of the points beyond the N'th in the sequence are bunched together and the least upper bound of all of these points is close to $\bar{a}$. To tie the proof together we just have to relate all of these points in the tail of the sequence to the least upper bound of these points and we are done.
- pick $K \geq N$ such that $d(a_N, x_K) < \epsilon/3$: we can do this for the following reason: a_N is the sup of $A_N = \{x_N, x_{N+1}, ...,\}$; by Theorem B, there's some point in A_N which is within $\epsilon/3$ of a_N; Let K denote the index of this point.

Now apply the triangle inequality which just formalizes the ideas that: the points in the sequence are bunched; the suprema are close to $\bar{a}$; the top end the sequence is close to the least upper bound.

[4] Recall from last lecture the The Axiom of Completeness: Every non-decreasing (non-increasing) sequence $\{x_n\}$ in $\mathbb{R}$ that is bounded above (below) converges to a point $x \in \mathbb{R}$ in the Euclidean metric.

Formally, for arbitrary $n > N$, we have

$$\begin{aligned} d(x_n, \bar{a}) &\leq d(x_n, x_K) + d(x_K, a_N) + d(a_N, \bar{a}) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \quad \Box \end{aligned}$$

Question: What part of the proof invokes the special property of the real line. That is, the theorem isn't true for Cauchy Sequences in general.

Answer: We utilize the fact that every nonincreasing, bounded sequence in $\mathbb{R}$ converges to a point in $\mathbb{R}$. In this case the nonincreasing sequence is the sequence of sups.

Question: The proof above relied critically on the fact that a Cauchy sequence is *bounded*. But the sequence $\{x_n\}$ defined by $x_n = n$, which is Cauchy w.r.t. the metric ρ, does not appear to be bounded. How would we prove the theorem for sequences like this and metrics like ρ.

Answer: This sequence *is* in fact bounded *w.r.t.* ρ. To make this precise, we need a more general notion of boundedness than the one we gave earlier: a set $S \subset X$ is *bounded* w.r.t. an arbitrary metric d, if there exists an element $b \in X$ such that for all $s \in S$, $d(s, 0) \leq d(b, 0)$.

To summarize this presentation on Cauchy sequences, for many students, it's really hard to appreciate the difference between a Cauchy sequence and a convergent sequence. At first sight, it seems that the definition of Cauchy is just another way of writing the definition of convergence. But as we've seen, while in some special situations, the two concepts are equivalent they aren't equivalent in general: convergentness is a strictly stronger property than Cauchyness.

To determine when the concepts are equivalent and when they are not you *have to consider the properties of the space that the sequence lives in.* If a sequence lives in a space that has no holes in it, then Cauchy sequences will be convergent; such spaces are called *complete* spaces. But as we've seen there are many incomplete spaces—the space of continuous functions, the rationals, for example. In such spaces, Cauchy sequences *may* be convergent, but they need not necessarily be.