ARE211, Fall 2009

Contents

1. Analysis (cont) 1
1.4. Convergence of Sequences 1
1.4.1. Boundedness of Sequences 2
1.4.2. Boundedness of Sequences (cont) 4
1.5. Least Upper bounds and Greatest Lower Bounds, Suprema and Infima 5

1. Analysis (cont)

1.4. Convergence of Sequences

Last lecture I introduced a number of metrics, the d^1, d^2 and d^∞ metrics. An important property of these metrics is that they are "all the same," in a sense that will be clearer in a minute. In this lecture, I'll refer to "the Euclidean" metric:" in fact lots of Euclidean metrics. Any one of the ones mentioned above will do, since they are all equivalent.

Definition: We say that a sequence $\{x_n\}$ in X *converges* to an element $x \in X$ in the metric d if $\forall \epsilon > 0$ $\exists N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x) < \epsilon$.

Definition: Two metrics d and ρ on X are said to be *equivalent* if a sequence $\{x_n\}$ in X *converges* to an element $x \in X$ in the metric d iff $\{x_n\}$ in X *converges* to an element $x \in X$ in the metric ρ.

As an exercise you should try to prove that for any n and m, the metrics $d^r(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^{n}(x_i - y_i)^r\right)^{1/r}$ and $d^q(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^{n}(x_i - y_i)^q\right)^{1/q}$ are equivalent.

A sequence that converges to a point is called a convergent sequence. The point that it converges to is called a limit of the sequence. (It's also *the* limit of the sequence, i.e., there's only one, but that's a result not a definition.)

Example: $\{x_n\}$ defined by $x_n = 1/n$ converges to zero in the Euclidean metric, but not in the discrete metric.

Proof: We need to show that for all ϵ, there exists N such that for all $n > N$, $d(x_n, 0) < \epsilon$. Now in the Euclidean metric, $d(x_n, 0) = \sqrt{(x_n - 0)^2} = |x_n|$, i.e., we just need to show that for n sufficiently large, $|x_n| < \epsilon$. Pick $N \geq 1/\epsilon$. Then $1/N < \epsilon$ and for $n > N$, $1/n < 1/N \leq \epsilon$. □

On the other hand, consider the discrete metric. To show that x_n *doesn't* converge to zero, we need to show that there exists $\epsilon > 0$ such that for all N, there exists $n > N$ such that $d(x_n, 0) > \epsilon$. Pick $\epsilon = 1/2$. Let N be arbitrary and observe that $d(x_{N+1}, 0) = 1 > \epsilon$. Done

Note that what we did here was an example of a general proof technique: to prove that an assertion involving $\forall$'s and $\exists$'s is false, go through and change the $\forall$'s to $\exists$ and the $\exists$'s to $\forall$.

Theorem: A sequence $\{x_n\}$ converges to $\alpha \in \mathbb{R}$ in the discrete metric if and only if there exists N such that for all $n > N$, $x_n = \alpha$.

1.4.1. *Boundedness of Sequences.*

Definition: A sequence is *bounded above* by $b \in \mathbb{R}$ if $x_n \leq b$, for all $n \in \mathbb{N}$. A sequence is *bounded above* if it is bounded above by *some* $b \in \mathbb{R}$. Defn of bounded below is parallel.

Similarly, a *set* X is *bounded above* by $b \in \mathbb{R}$ if $x \leq b$, for all $x \in X$.

Example: The sequence $\{x_n\}$ defined by $x_n = (-1)^n$ is bounded above by 1 and below by -1. (It's bounded above by lots of other things too, e.g, 2, π, etc, and similarly bounded below by lots of other things. The point is that in order to be bounded above, a sequence just has to be bounded above by *something*).

The Axiom of Completeness: Every non-decreasing sequence $\{x_n\}$ in $\mathbb{R}$ that is bounded above converges to a point $x \in \mathbb{R}$ in the Euclidean metric. Every non-increasing sequence $\{x_n\}$ in $\mathbb{R}$ that is bounded below converges to a point $x \in \mathbb{R}$ in the Euclidean metric. (Not a theorem! Can't prove it. Have to assume it.)

Intuition for these is straightforward: take the latter statement; you have an infinite number of points that keep declining, or at least doesn't increase; but there's a floor to the decline: the points have to accumulate, because there's nowhere else for them to go.

The Axiom is a very special property that depends both on the set $\mathbb{R}$ and the Euclidean metric. To see that it isn't true for general sets, consider the set of *rationals*, denoted $\mathbb{Q}$, which consists of all ratios of integers. If you replaced $\mathbb{R}$ by $\mathbb{Q}$ in the above statement, it would obviously be false: consider a non-increasing sequence of rational numbers which, when viewed as real numbers, converges to the non-rational number π. (An example of such a sequence is given below in green.) Since it's non-increasing, it's bounded below by π. But if you replaced the symbol $\mathbb{R}$ everywhere by $\mathbb{Q}$ in the statement of the Axiom above, it would clearly be nonsense. Similarly, the Axiom would be equally nonsensical if the words "Euclidean metric" were replaced by "discrete metric."

The above highlights a special property of the real numbers, which is that they are *complete*, i.e., have no holes in them. (Again, this is an assumption, it is not a result that can be proved by invoking other properties of the real line.)

Example: Consider the following sequence $\{1, \frac{-1}{3}, \frac{1}{5}, \frac{-1}{7}, \frac{1}{9}, ...\}$ and let y_n denote the n'th element of this sequence. Now define the new sequence $\{x_n\}$ by: $x_n = 4\sum_{k=1}^{2n-1} y_n$. (e.g., $x_2 = 4(1 - 1/3 + 1/5)$, $x_3 = x_2 + 4(-1/7 + 1/9)$, etc. Note that it is a non-increasing sequence (in fact strictly decreasing). Also, note that the entire sequence lives in the set of *rationals*, denoted by $\mathbb{Q}$, which consists of all ratios of integers. Now viewed as a sequence living in $\mathbb{R}$, the sequence x_n converges (believe it or not) to $\pi/4$. But viewed as a sequence living in $\mathbb{Q}$, it is a non-convergent sequence, since its "limit"—if it had one—can only be $\pi/4$, but $\pi/4 \notin \mathbb{Q}$.

To see that the Axiom of Completeness is in fact a highly non-trivial assumption, consider the set of continuous functions $C^0 = \{f : [0,1] \to \mathbb{R}\}$. Intuitively, it's fairly obvious what a non-decreasing sequence in C^0 would look like. For example, here's one: the sequence of continuous functions $\{f_1, f_2, ...f_n...\}$, where $f_n(x) = \begin{cases} nx & \text{if } 0 \leq x < 1/n \\ 1 & \text{if } 1/n \leq x \leq 1 \end{cases}$.

While I'm not going to define metrics for functions (you need one for the domain and one for the range), it's intuitive that for most such metrics (not all, though), the sequence $\{f_n\}$ will *not* converge to any continuous function. So the Axiom of Completeness would be violated if "$\mathbb{R}$" in the specification above were replaced by "C^0".

1.4.2. *Boundedness of Sequences (cont).* The following questions came up in class in past years.

Question: Given a sequence $\{x_n\}$, suppose that $|x_{n+1} - x_n|$ converges to zero. Can we conclude that the sequence is bounded?

Answer: *No.* Consider the sequence $\{x_n\}$ defined by $x_n = \sum_{k=1}^{n} 1/k$. Observe that difference between x_{n+1} and x_n is $1/(n+1)$. However, the sequence $\{1/x_n\}$, i.e., $\{\sum_{k=1}^{n} 1/k\}$ doesn't converge. (This fact is isn't at all obvious (see Berck and Sydsaeter, 7.10 for verification that it is indeed true).)

Question: Can a sequence converge to different things in different metrics?

Answer: Not if the sequence belongs to a finite dimensional space, i.e., $\mathbb{R}^n$, but more generally yes. For sequences in $\mathbb{R}^n$, all metrics are "equivalent" in the sense that if a sequence in $\mathbb{R}^n$ converges in two distinct metrics, then it converges to the same point in both metrics. In *infinite dimensional spaces* (we'll learn about them in Linear Algebra), however, this is not true. Jeff La France and I came up with an example, using a sequence of *continuous functions* similar to the ones we defined in a previous lecture, $\{f_1, f_2, ...f_n...\}$, where $f_n = \begin{cases} -1 & \text{if } x \leq -1/n \\ nx & \text{if } -1/n < x < 1/n \\ 1 & \text{if } x \geq 1/n \end{cases}$. Depending on your metric, this sequence can converge to either

$$\bar{f} = \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases} \text{ or } \bar{g} = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases} \text{ or } \bar{h} = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}.$$

1.5. **Least Upper bounds and Greatest Lower Bounds, Suprema and Infima**

Every finite set has a largest and a smallest element. However, this may not be true of infinite sets. E.g., the set $1 - 1/2, ..., 1 - 1/n, ...$ has no highest element. It "nearly" gets to 1 however.

So for bounded sets, we define something called its *least upper bound* which is "almost the maximum," the point that would be the maximum if the set had a maximum. In the above example the least upper bound of the set is 1. Similarly, the least upper bound of the set $(\underline{x}, \bar{x})$ is $\bar{x}$.

Definition: Let $S \subset \mathbb{R}$. A number $b \in \mathbb{R}$ is an *upper bound for* S if $s \leq b$, for all $s \in S$. A number b is the *least upper bound for* S if b is an upper bound for S and if $b \leq b'$, for any upper bound b' of S.

Analogously, we have *greatest lower bound* of a set, which is "almost the minimum," the point that would be the minimum if the set had a minimum. In the above example the greatest lower bound of the set is $1/2$, which is also the minimum of the set. Similarly, the greatest lower bound of the set $(\underline{x}, \bar{x})$ is $\underline{x}$.

Naturally if a set *does indeed* have a maximum then the least upper bound of the set is that maximum, e.g., the greatest lower bound and the least upper bound of the set $[\underline{x}, \bar{x}]$ are $\underline{x}$ and $\bar{x}$.

Definition: Let $S \subset \mathbb{R}$. A number $b \in \mathbb{R}$ is a *lower bound for* S if $s \geq b$, for all $s \in S$. A number b is a *greatest lower bound for* S if b is a lower bound for S and if $b \geq b'$, for any lower bound b' of S.

If a set has a least upper bound, it is also called the *supremum* of the set. Similarly, if a set has a greatest lower bound, it is also called the *infimum* of the set. Some sets, however, don't have a least upper bound (e.g., $\mathbb{R}, \mathbb{N}$). In this case, we say that the supremum of the set is ∞. Similarly, if a set doesn't have a greatest lower bound, we say that it's infimum is $-\infty$.

Notation: $\sup(S)$ and $\inf(S)$ denote, respectively, the supremum and infimum of the set S.

A critical property of the real line $\mathbb{R}$ is:

Theorem: Every nonempty set $S \subset \mathbb{R}$ that has an upper bound has a least upper bound. Every nonempty set $S \subset \mathbb{R}$ that has an lower bound has a greatest lower bound in $\mathbb{R}$.

Note the difference between this theorem and the Axiom of Completeness. The latter was about sequences, this is about sets. It is, however, another way of saying that the real line doesn't have any "holes" in it. It is, clearly, very closely related to the Axiom of Completeness. In fact, we could have made this the axiom and then turned the Axiom of Completeness into a theorem.