ARE211, Fall 2009

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Contents

1. Analysis 1

1.1. References 1

1.2. Sequences 2

1.3. Distance/Metrics 3

1. Analysis

Heavy emphasis on proofs in this section. Remember that it was the proofs that killed last year's class, and there was no quick way of helping them to do them. So we'll do a lot of them this semester. The best topic to learn how to proofs in is analysis.

1.1. **References**

Chapter 12 in Simon-Blume

Chapter 1 and 2: Elementary Classical Analysis, by J. Marsden (on Reserve)

1.2. **Sequences**

The *natural numbers*, denoted $\mathbb{N}$, are 1,2,3,4 ..., going on for ever.

A *sequence* is a mapping from the natural numbers to a set S, i.e., $f : \mathbb{N} \to S$; $f(n)$ is the n'th element of the sequence. Typically, we suppress the functional notation: instead of writing the image of n under f as $f(n)$ we denote it by x_n and write the sequence as $\{x_1, x_2, ..., x_n, ...\}$, i.e., $f(n) = x_n$.

A collection $\{y_1, y_2, ..., y_n, ...\}$ is a *subsequence* of another sequence $\{x_1, x_2, ..., x_n, ...\}$ if there exists a *strictly increasing* function $\tau : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, $y_n = x_{\tau(n)}$. Note that τ maps the *domain* of the subsequence into the *domain* of the original sequence. For example, consider the sequence $\{3, 6, ..., 3n, ...\}$ and the subsequence $\{6, 12, ..., 6n, ...\}$. In this case, the function τ we need is $\tau(n) = 2n$, i.e., for all n, $y_n = x_{2n}$: e.g., $y_1 = x_2 = 6$, $y_2 = x_4 = 12$. That is, you construct a subsequence by throwing out elements of the original sequence, but keeping an infinite number of the original elements *and preserving their order.*

It's useful to compare the following kinds of mappings. The only distinction between them is the domain of the mapping.

(1) $v : \{1, ..., N\} \to \{1\}$, This is more commonly thought of as an N-*vector* of ones, i.e., $(1, ..., 1)$.
(2) $x : \mathbb{N} \to \{1\}$. This is a *sequence* of ones, i.e., $x_n = 1$, for all n.
(3) $f : \mathbb{R}_+ \to \{1\}$. This is a continuous *function*, mapping the non-negative real numbers to 1, i.e., $f(\cdot) = 1$.

Until you're taught to think otherwise, you'd probably think of only the latter as a real "function." Actually, all three mappings satisfy the true definition of a function, i.e., each of them assigns a *unique* point in the range to each point in its respective domain.

Some examples of sequences:

(1) $\{1, 2, 3, 4...\}$

(2) $\{1, 1/2, 4, 1/8...\}$

(3) $\{-1, 1, -1, 1...\}$

(4) $\{1, 1/2, 1/3, 1/4...\}$

(5) sequences aren't necessarily maps from $\mathbb{N}$ into scalars. We could have a map from $\mathbb{N}$ into the set of continuous functions. For example, consider the sequence of continuous functions

$$\{f_1, f_2, ...f_n...\}, \text{ where } f_n = \begin{cases} -1 & \text{if } x \leq -1/n \\ nx & \text{if } -1/n < x < 1/n \\ 1 & \text{if } x \geq 1/n \end{cases}$$

(6) sequences don't necessarily have a *closed form representation*, i.e., you can't necessarily write down a formula that expresses the sequence. For example, if you started generating random numbers and continued forever, you would have a sequence.

Note that the difference between a *sequence* and a *set* of comparable size is that the *order* of the elements in a sequence matters, while the order of the elements of a set does not. Thus, the sequences $\{1, 2, 3, 4...\}$ and $\{2, 1, 3, 4...\}$ are different, while the sets $\{1, 2, 3, 4...\}$ and $\{2, 1, 3, 4...\}$ are the same. Moreover, the set $\{1, 1, 1, 1...\}$ is just the singleton set $\{1\}$, while the sequence $\{1, 1, 1, 1...\}$ is quite different from the scalar 1.

1.3. **Distance/Metrics**

Analysis is all about how close things are to each other. Does a sequence converge to a point? There are lots of notions of closeness in mathematics, some of them more intuitive than others. So long as we are considering closeness in the context of Euclidean space, most notions of closeness are essentially equivalent. However, once we get to consider closeness in the context of *functions*, there is a vast variety of quite different notions. Mathematicians have an abstract notion of what is a legitimate measure of closeness.

Definition: : a *metric* or *distance function* on a set S is a function $d : S \times S \to \mathbb{R}$ satisfying, for all $x, y \in S$:

(1) $d(x,y) = d(y,x)$ (*symmetry*)
(2) $d(x,y) \geq 0$ (*nonnegativity*)
(3) $d(x,y) = 0$ iff $x = y$ (two elements are a positive distance apart iff they are different from each other)
(4) $d(x,y) \leq d(x,z) + d(z,y)$, for all $z \in S$ (*the triangle inequality*)

The last property of a metric is the one that has the most bite, and the one that really captures the spirit of distance: it states that the shortest distance between two points is a straight line.

Examples of metrics

(1) on $\mathbb{R}$: $d^1(x,y) = |x - y|$.
(2) on $\mathbb{R}^n$: $d^2(\mathbf{x},\mathbf{y}) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$
(3) on $\mathbb{R}^n$: $d^\infty(\mathbf{x},\mathbf{y}) = \max\{|x_i - y_i| : i = 1, ..., n\}$.
(4) on $\mathbb{R}^n$: $d(\mathbf{x},\mathbf{y}) = \begin{cases} 1 & \text{if } \mathbf{x} \neq \mathbf{y} \\ 0 & \text{if } \mathbf{x} = \mathbf{y} \end{cases}$ (we'll call this the *discrete* metric).

An example of a function that is *not* a metric is $e(\mathbf{x},\mathbf{y}) = \min\{|x_i - y_i| : i = 1, ..., n\}$.

Let's check that the function $d(\mathbf{x},\mathbf{y}) = \begin{cases} 1 & \text{if } \mathbf{x} \neq \mathbf{y} \\ 0 & \text{if } \mathbf{x} = \mathbf{y} \end{cases}$ is indeed a metric. It clearly satisfies the first three properties. What about the triangle inequality. First observe that if $\mathbf{x} = \mathbf{y}$, then $d(\mathbf{x},\mathbf{y}) = 0$.

Since $d(\mathbf{x},\mathbf{z}) + d(\mathbf{z},\mathbf{y})$ is necessarily nonnegative, then inequality holds. Now suppose that $\mathbf{x} \neq \mathbf{y}$ so that $d(\mathbf{x},\mathbf{y}) = 1$. In this case, for all $\mathbf{z}$, either $\mathbf{z} \neq \mathbf{x}$ or $\mathbf{z} \neq \mathbf{y}$ in which case either $d(\mathbf{x},\mathbf{z})$ or $d(\mathbf{z},\mathbf{y})$ is 1 so the inequality is satisfied.

Now let's check that the function $e(\mathbf{x},\mathbf{y}) = \min\{|x_i - y_i| : i = 1, ..., n\}$ is *not* a metric. Well it fails the third condition, since $e((1,1),(1,2)) = 0$, but $(1,1) \neq (1,2)$. More fundamentally it fails the last condition: set $\mathbf{x} = (1,1)$, $\mathbf{y} = (2,2)$, $\mathbf{z} = (1,2)$, $e((1,1),(2,2)) = 1$ but $e((1,1),(1,2)) = e((1,2),(2,2)) = 0$ so that $d(\mathbf{x},\mathbf{y}) > d(\mathbf{x},\mathbf{z}) + d(\mathbf{z},\mathbf{y})$.

When S is a space of *functions*, condition 3 is too restrictive. In economics, for example, we often encounter functions that *aren't* equal to each other, but are said to be of distance zero from each other. In particular, it is often natural to say that the distance between two functions is the *integral* of the absolute value of the difference between them. But if two functions differ at only a finite (indeed countable) number of points, then the integral of the absolute value of the difference between them will be zero.

Condition 3 is inconsistent with this usage. Accordingly, we define a function to be a *pseudo-metric* if it satisfies all axioms except 3. E.g., if S is the set of integrable functions mapping $\mathbb{R}$ to $\mathbb{R}$, then the function $\rho : S \times S \to \mathbb{R}$ defined by $\rho(f,g) = |\int f dx - \int g dx|$ is a pseudo-metric but not a metric.

(1) To see that ρ is *not* a metric, consider the function f_1 defined above on p.3 as example 5. Because the function is so symmetric, clearly $\int f_1 dx = \int -f_1 dx = 0$, so that $\rho(f_1, -f_1) = 0$, but these functions are not equal.
(2) On the other hand, to see that ρ is a pseudo-metric, observe that it's obviously symmetric and non-negative. The only thing remaining to check is that it satisfies the triangle inequality. To prove this, we use the following Lemma

Lemma: for any $x, y \in \mathbb{R}$, $|x| + |y| \geq |x + y|$.

Proof of the Lemma: It's obvious that if x and y both have the same sign then $|x| + |y| = |x + y|$. Now suppose without loss of generality (w.l.o.g.) that $x \geq 0 > y$. In this case,

$$|x| + |y| \quad = \quad x + (-y) \quad > \quad x \quad > \quad |x - (-y)| \quad = \quad |x + y| \qquad \square$$

We can now check that ρ satisfies the triangle inequality. For any functions $f, g, h \in S$,

$$\rho(f,h) + \rho(h,g) \quad = \quad \left| \int f dx - \int h dx \right| + \left| \int h dx - \int g dx \right|$$

which from the lemma is

$$\begin{aligned} &\geq \left| \int f dx - \int h dx + \int h dx - \int g dx \right| \\ &= \left| \int f dx - \int g dx \right| \\ &= \rho(f,g) \end{aligned}$$