L Karp
Open economy microeconomics
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Solution Problem Set 6

1) In general it might be optimal to close down production; in economically uninteresting cases, it might be optimal (for this formulation of the problem) to use an infinite *subsidy.* I will assume throughout that neither of these cases arises. In other words, I assume that the optimal solution is interior: the tax is greater than $-\infty$ (i.e. it is not an infinite subsidy) and it is small enough that production is positive. Given this assumption of an interior equilibrium, and in view of the differentiability of all functions, the first order condition is necessary for an optimum.

Using the equilibrium condition $D = S$, the first order condition is

$$(\alpha S - S + (t - \gamma) S') \frac{dP}{dt} = 0$$

which simplifies to equation (2).

Notice that when $\alpha > 1$ the tax is lower than the Pigouvian tax of γ, and that a lower tax leads to a higher equilibrium price (as we would expect). Also, the optimal tax is negative if

$$\frac{\eta\gamma}{\alpha - 1} < P.$$

This inequality holds if α is sufficiently large or γ sufficiently small. If the regulator attaches enough importance to producers' welfare, or if the externality is small enough, the regulator subsidizes production (and therefore subsidizes consumption in the closed economy).

At this level of generality, we are not certain that the maximand is globally concave. From the first order condition above, you can see that the second order condition involves the second derivative of S and the term $\frac{d^2P}{dt^2}$. I can write down the second order condition, but I can already see that it will be an ugly mess, and will almost certainly not be informative. Therefore, I would like to find a different approach to guarantee that the necessary condition is also sufficient.

We can get a bit of insight using graphical methods. If the producer price is P and the tax is given by equation (2) in the problem set, the consumer price is

$$\hat{P} = P + t = \frac{1 - \alpha + \eta}{\eta} P + \gamma.$$

Note that $\frac{d\hat{P}}{dP} \gtrless 0$ as $1 - \alpha + \eta \gtrless 0$. Since η is a function of P, the sign of this expression can change. However, I will consider the simpler case were the sign does not change. I can write the equilibrium condition as

$$D(\hat{P}(P)) = S(P).$$

Figures 1 and 2 graph the demand function $D(P)$ and the function $D(\hat{P}(P))$. (I have drawn these as linear, but this is only a convenience – I do not assume linearity.) These two functions intersect at $P^* \equiv \frac{\gamma\eta}{\alpha - 1}$, the price at which $\hat{P} = P$. The two figures show the two cases, where $1 - \alpha + \eta \gtrless 0$. The figures also contain the graph of $D(P) - \gamma$. (I would use that graph to obtain the optimal tax in the case where $\alpha = 1$.) The solution to the necessary condition occurs where the supply function (not shown) intersects the graph of $D(\hat{P}(P))$. This intersection must occur above the graph of $D(P) - \gamma$ since I know from the necessary condition that $t < \gamma$.

The supply function has positive slope. For the case where $1 - \alpha + \eta > 0$ (Figure 1) there is a unique solution to $D(\hat{P}(P)) = S(P)$; that is, there is a unique solution to the necessary condition, so this condition is also sufficient (given my assumption of an interior optimum).

However, for the case where $1 - \alpha + \eta < 0$ (Figure 2) there could be multiple intersections, which alternate between local maxima and local minima. Thus, when $1 - \alpha + \eta < 0$ I do not know whether a tax that satisfies the necessary condition is a local min or a local max.

Therefore, I will assume (for purpose of Question 2) that $1 - \alpha + \eta > 0$. This assumption might be restrictive, in that it might exclude interesting possibilities. If I were writing a paper on this problem I would do some numerical experiments that would allow me to drop this assumption. This assumption puts a restriction on the magnitude of α. However, the assumption is consistent with the tax being either positive or negative. In figure 1, the supply function (not shown) could intersect the graph of $D(\hat{P}(P))$ above or below the price $\frac{\gamma\eta}{\alpha - 1}$, resulting in a negative or a positive producer tax.

2) (Part b) In the open economy, with a production tax, the consumer price is P^w and the producer price is $P = P^w - t$. The regulator's objective

is

$$V(t;\alpha) = \alpha \int_{\underline{p}}^{P^w - t} S(p)dp + \int_{P^w}^{\bar{p}} D(p)dp + (t - \gamma)\, S(P).$$

The first order condition is

$$-\alpha S + S - (t - \gamma)\, S' = 0$$

which implies

$$\begin{aligned} t &= \frac{(1-\alpha)\, S}{S'} \frac{P}{P} + \gamma \\ t &= \frac{1-\alpha}{\eta} P + \gamma. \end{aligned}$$

The formulae for the optimal tax under the open and closed economies have the same form, but in general the equilibrium prices are different in the two cases. Using $P = P^w - t$, write t as a function of P^w:

$$\begin{aligned} t &= \frac{1-\alpha}{\eta} (P^w - t) + \gamma \\ t &= \frac{(1-\alpha)\, P^w + \gamma\eta}{\eta + 1 - \alpha} \end{aligned}$$

(Part c) Using the results from Questions 1 and 2b, write the tax in the closed economy as

$$t^c = \frac{1-\alpha}{\eta} P^c + \gamma$$

and in the open economy as

$$t^o = \frac{1-\alpha}{\eta} P^o + \gamma = \frac{(1-\alpha)\, P^w + \gamma\eta}{\eta + 1 - \alpha}$$

where P^c and P^o are the equilibrium producer prices in the closed and open economies, respectively. Opening a closed economy reduces the tax if and only if

$$\begin{aligned} t^c &> t^o \Longleftrightarrow \\ \frac{1-\alpha}{\eta} (P^c - P^o) &> 0 \Longleftrightarrow \\ P^c &< P^o \end{aligned}$$

That is, opening a closed economy reduces the tax if and only if the producer price in the open economy is higher than the producer price in the closed economy.

Of course, both of these prices depend on the tax, so this necessary and sufficient condition is not expressed in terms of the primitives of the model (the supply and demand functions and the parameters α and P^w.) An alternative (and possibly more revealing) way of writing the necessary and sufficient condition is as follows:

$$\begin{aligned}
t^c &> t^o \Longleftrightarrow \\
\frac{1-\alpha}{\eta}P^c + \gamma &> \frac{(1-\alpha)\,P^w + \gamma\eta}{\eta+1-\alpha} \Longleftrightarrow \\
\frac{1-\alpha}{\eta}P^c &> \frac{P^w - P^w\alpha - \gamma + \gamma\alpha}{\eta+1-\alpha} \Longleftrightarrow \\
P^c &< \frac{P^w - P^w\alpha - \gamma + \gamma\alpha}{\eta+1-\alpha}\frac{\eta}{1-\alpha} = \eta\frac{P^w-\gamma}{\eta+1-\alpha}
\end{aligned}$$

Here is where I will use the assumption that $\eta + 1 - \alpha > 0$. This assumption enables me to write the implications above as

$$\begin{aligned}
t^c &> t^o \Longleftrightarrow \\
P^c\frac{\eta+1-\alpha}{\eta} + \gamma &< P^w
\end{aligned}$$

In the closed economy the *consumer price* is $P^c + t^c = P^c\frac{\eta+1-\alpha}{\eta} + \gamma$. In the open economy (that uses a production tax) the consumer price is P^w. Thus, we can compare the open and closed economy taxes by comparing the equilibrium consumer price in the closed economy with the world price.

Supposed that we have a closed economy that is currently imposing the optimum (closed economy) producer tax. The economy is deciding whether to open up to free trade, given a (fixed) world price P^w. Of course, if it opens up to trade, there is a change in the optimal tax. However, we can perform the thought experiment, "open up to trade and maintain the current tax, t^c". Given the tax t^c, the equilibrium (inverse) supply curve is $S + t$, graphed in Figure 3. This figure shows the case where $t > 0$, but this fact is not important. The equilibrium consumer price is shown as $P^c\frac{\eta+1-\alpha}{\eta} + \gamma = \hat{P}^*$. If $P^w > \hat{P}^*$, the country would be an exporter at the initial tax t^c.

In summary, my conclusion is that if the country would become an exporter at the initial tax, then when the country opens up to trade, it will reduce the tax. If the country would become an importer at the initial tax, then when it opens up to trade, it will want to increase the tax. (Remember that this conclusion is based on the maintained assumption that $\eta + 1 - \alpha > 0$.)

3) In the open economy when pollution is caused by consumption and the regulator uses a consumption tax, the objective is

$$V(t;\alpha) = \alpha \int_{\underline{p}}^{P^w} S(p)dp + \int_{P^w+t}^{\bar{p}} D(p)dp + (t-\gamma)\, D(P^w+t).$$

The first order condition is

$$-D(P^w+t) + D(P^w+t) + (t-\gamma)\, D'(P^w+t) = 0$$

which implies

$$t = \gamma.$$

4) The first order condition from maximizing equation (5) can be simplified to

$$\left((\alpha-1) + (\alpha t - \gamma)\frac{\eta}{P}\right)\frac{dP}{dt} + \alpha - 1 = 0.$$

By differentiating the equilibrium condition $S(P) = D(P+t)$ we obtain

$$\frac{dP}{dt} = \frac{-\rho}{\eta\left(1+\frac{t}{P}\right)+\rho}.$$

Write the first order condition as

$$\left((\alpha-1) + (\alpha t - \gamma)\frac{\eta}{P}\right)\left(\frac{-\rho}{\eta\left(1+\frac{t}{P}\right)+\rho}\right) + \alpha - 1 = 0$$

and solve for t to obtain

$$t = \frac{\rho\gamma + (\alpha-1)\,P}{\alpha\,(\rho-1)+1}.$$

The consumer price is

$$\hat{P} = P + t = P + \frac{\rho\gamma + (\alpha-1)\,P}{\alpha\,(\rho-1)+1} = \rho\frac{P\alpha+\gamma}{\alpha\rho-\alpha+1}$$

I will use the same graphical technique as in Question 1. I can write the equilibrium condition as $D(\hat{P}(P)) = S(P)$. Figures 4 and 5 show the graph of $D(P)$ and of $D(\hat{P}(P))$ for two cases Note that $P = \hat{P}$ implies that $P = -\rho\frac{\gamma}{\alpha-1} < 0$. Thus, the graphs of $D(P)$ and of $D(\hat{P}(P))$ do not intersect for positive P. Again I need to consider two cases.

First consider the case $\alpha\rho - \alpha + 1 > 0$, so $\frac{d}{dP}D(\hat{P}(P)) = D'\frac{\rho\alpha}{\alpha\rho-\alpha+1} < 0$ so the graph of $D(\hat{P}(P))$ has a negative slope. In addition

$$\begin{aligned} \hat{P} &\equiv \rho\frac{P\alpha+\gamma}{\alpha\rho-\alpha+1} > P \Longleftrightarrow \\ \rho\left(P\alpha+\gamma\right) - P\left(\alpha\rho-\alpha+1\right) &> 0 \Longleftrightarrow \\ \rho\gamma + \left(\alpha-1\right)P &> 0 \Longleftrightarrow \\ \left(\alpha-1\right)P &> -\rho\gamma. \end{aligned}$$

The last inequality is always true, so for $\alpha\rho - \alpha + 1 > 0$ I know that $D(\hat{P}(P)) < D(P)$ as shown in Figure 4. (Again, I am making no assumption about the shape of the demand curve; I have drawn it as linear only for convenience.) The equilibrium producer and consumer prices are shown as P^c and $\hat{P}$ in the figure. There is a unique solution to the necessary condition $D(\hat{P}(P)) = S(P)$, so given my assumption of an interior optimum, I know that the necessary condition is sufficient. Note also that at this equilibrium, $t > 0$: it is optimal to tax production.

Now consider the case where $\alpha\rho - \alpha + 1 < 0$, so $\frac{d}{dP}\hat{P} < 0$. In addition $\hat{P} > 0 \Longleftrightarrow P < -\rho\frac{\gamma}{\alpha-1} < 0$. However, the function D is defined only for positive arguments (a positive price). Therefore, there is no price at which both the quantities $D(\hat{P}(P)$ and $D(P)$ are positive. Consequently there cannot be an interior optimum for the case $\alpha\rho - \alpha + 1 < 0$.

If you are not convinced, consult Figure 5. The functions $D(\hat{P}(P)$ and $D(P)$ intersect at $P < 0$ and they have opposite slopes. Suppose that I have a function $S(P)$ that intersects $D(\hat{P}(P)$ as shown in figure 5. The candidate equilibrium sales level is shown as Q^c. However, there is no positive price that induces consumes to purchase this level – so even an infinite subsidy could not support the candidate equilibrium.

I conclude that in this model there is an interior maximum if and only if $\alpha\rho - \alpha + 1 > 0$

5) (Part a) The objective is

$$V(t;\alpha) = \alpha\left(\int_{\underline{p}}^{P^w - t} S(p)dp + tS(P^w - t)\right) + \int_{P^w}^{\bar{p}} D(p)dp + -\gamma S(P^w - t).$$

The first order condition is

$$t = \frac{\gamma}{\alpha}.$$

Using this result and the result from question 4, write the taxes in the open and closed economies as

$$t^c = \frac{\rho\gamma + (\alpha - 1) P}{\alpha(\rho - 1) + 1} \qquad t^o = \frac{\gamma}{\alpha}$$

which implies

$$\begin{aligned} t^o - t^c &< 0 \Longleftrightarrow \\ -(\alpha - 1)\frac{P\alpha + \gamma}{\alpha(\alpha\rho - \alpha + 1)} &< 0 \Longleftrightarrow \\ (\alpha\rho - \alpha + 1) &> 0. \end{aligned}$$

In Question 4 the last inequality has to hold at an interior maximum. With this restriction, we have shown that opening the closed economy necessarily reduces the optimal tax.

(b) The case where the regulator returns the tax/quota revenue to consumers is identical to the case where the regulator values consumer and taxpayer utility equally – the case that we considered in Questions 1 -3. We noted in Question 3 that in the case were the damage is caused by consumption, opening the closed economy always increases the tax.