# Two-Period Renewable Resources Model
# With Non-Zero Interest Rate

· suppose we want to maximize net present value (NPV)
· suppose that there is *not* open access to the resource
· do not assume, prior to solving the model, that we are in steady-state.
    -- it may or may not be the solution

In general, we would solve the problem over the "**planning horizon**" of our firm or agency.

A planning horizon is the number of time periods into the future for which the firm or agency makes plans.

To keep the mathematics manageable, we will focus on a two time period planning horizon.

If demand is very low or harvesting costs are very high, then stock may remain at the end of the planning horizon.

A **salvage value function** gives the **value of any remaining stock at the** end of the planning horizon.

Note: salvage value is a function of stock, not harvest. In the case of a fishery, what would be sold at the end of the planning horizon is not caught fish (i.e., not "X"), but rather the right to catch the remaining fish in the sea, (i.e., the resource asset "S").

# Two-period Model (cont.)

| initial stock (stock at beginning of period 0) | stock at beginning of period 1 | final stock (stock at end of period 1) |
|---|---|---|

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------+---------------------------------+-----------------------+----
>      time
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So S1 S2

X0 X1

**Definitions:**

- t: time period  t = 0,1,2
- $S_t$: resource stock at time t
- $g(S_t)$: the growth function of the resource stock
- $X_t$: harvest at time t
- $B(X_t)$: total benefits from harvest at time t
- $C(X_t, S_t)$: total costs of harvest at time t
- $F(S_2)$: salvage value function
- r: interest rate

The **objective** is to maximize the net present value of harvest in period 0, harvest in period 1 and salvage value in period 2. The **choice variables** are X0, X1, S0 and S1.

$$\max_{X_0, X_1, S_1, S_2} NPV = \underbrace{B(X_0) - C(X_0, S_0)}_{\text{net benefit period 0}} + \underbrace{\frac{B(X_1) - C(X_1, S_1)}{1+r}}_{\text{discounted net benefit period 1}} + \underbrace{\frac{F(S_2)}{(1+r)^2}}_{\text{discounted "salvage value" of final stock}}$$

subject to:

(1) $g(S_0) = S_1 - S_0 + X_0$, equation of motion between periods 0 and 1

(2) $g(S_1) = S_2 - S_1 + X_1$, equation of motion between periods 1 and 2

Also, So is the given, initial stock.

# Two-period Model (cont.)

The **Lagrangian expression for this problem** is:

$$\max_{\substack{X_0, X_1 \\ S_1, S_2 \\ \lambda_0, \lambda_1}} L = B(X_0) - C(X_0, S_0) + \frac{B(X_1) - C(X_1, S_1)}{1+r} + \frac{F(S_2)}{(1+r)^2}$$

$$+\lambda_0 \left(g(S_0) - S_1 + S_0 - X_0\right) + \frac{\lambda_1}{1+r}\left(g(S_1) - S_2 + S_1 - X_1\right)$$

where $\lambda_0$ and $\lambda_1$ are Lagrange multipliers.

**FOC's**

(1) $$\frac{dL}{dX_0} = B_{X_0}(X_0) - C_{X_0}(X_0, S_0) - \lambda_0 = 0$$

-- MB(0) - MC(0) - user cost(0) = 0

(2) $$\frac{dL}{dX_1} = \frac{B_{X_1}(X_1)}{1+r} - \frac{C_{X_1}(X_1, S_1)}{1+r} - \frac{\lambda_1}{1+r} = 0$$

-- NPV(MB(1)) - NPV(MC(1)) - NPV(user cost(1)) = 0

(3) $$\frac{dL}{dS_1} = \frac{-C_{S_1}(X_1, S_1)}{1+r} - \lambda_0 + \frac{\lambda_1}{1+r}\left(\frac{dg}{dS_1} + 1\right) = 0$$

$$\text{rearranging}: \lambda_0 = \frac{-C_{S_1}(X_1, S_1)}{1+r} + \frac{\lambda_1}{1+r} + \frac{\lambda_1}{1+r}\left(\frac{dg}{dS_1}\right)$$

| | | | | | | |
|---|---|---|---|---|---|---|
| -- user cost in period 0 | = | NPV(increase in harvesting cost) | + | NPV(user cost associated with having one less unit of stock | + | NPV (user cost of having less growth in period 1 |

# Two-period Model (cont.)

**FOCs**:

(4) $$\frac{dL}{dS_2} = \frac{F_{S_2}(S_2)}{(1+r)^2} - \frac{\lambda_1}{1+r} = 0$$

-- NPV (marginal salvage value) - NPV (user cost(1)) = 0

(5) $$\frac{dL}{d\lambda_0} = g(S_0) - S_1 + S_0 - X_0 = 0$$

-- equation of motion between periods 0 and 1 must be satisfied.

(6) $$\frac{dL}{d\lambda_1} = g(S_1) - S_2 + S_1 - X_1 = 0$$

-- equation of motion between periods 1 and 2 must be satisfied.

· keeping a unit of stock unharvested has three effects:

(1) Loss of interest from not harvesting the stock today.

(2) Savings in extraction cost, because $C_S(Xt,St) < 0$.

(3) Additional growth of the resource stock. The present value of the additional growth in stock is $\frac{\lambda_1}{1+r}\left(\frac{dg}{dS_1}\right)$.

--Note: dg/dS can be positive or negative depending on whether S is < or > maximum sustainable stock

## Two-period Model (cont.)

In steady state, the shadow value of the resource remains constant:

$$\lambda_{t+1} - \lambda_t = 0$$

and MC of delayed harvest (lost interest) = MB of delayed harvest (growth + cost savings):

$$r\lambda_t = X_t c_S(S_t) + g_S(S_t)\,\lambda_t$$

The marginal benefit of delaying the harvest is the sum of reduced harvesting cost ($X_t\, c_S(S_t)$) plus the value of the added growth of the stock, $g_S(S_t)\,\lambda_t$.

Thus, at steady state,

$$\left[r - g_S(S_t)\right]\lambda_t = X_t\; c_S(S_t). \qquad \textbf{(7)}$$

Let $P_t$ be optimal price, i.e., $P_t = B_X(X_t)$:

At an optimal resource allocation, from FOC (2):

$$B_X(X_t) = P_t = \underbrace{\lambda_t}_{\text{user cost}} + \underbrace{c(S_t)}_{\text{extraction cost}} \qquad \textbf{(8)}$$

Since at steady state,

$S_{t+1} - S_t = 0$ and $\lambda_{t+1} - \lambda_t = 0$, then $P_{t+1} - P_t = 0$.

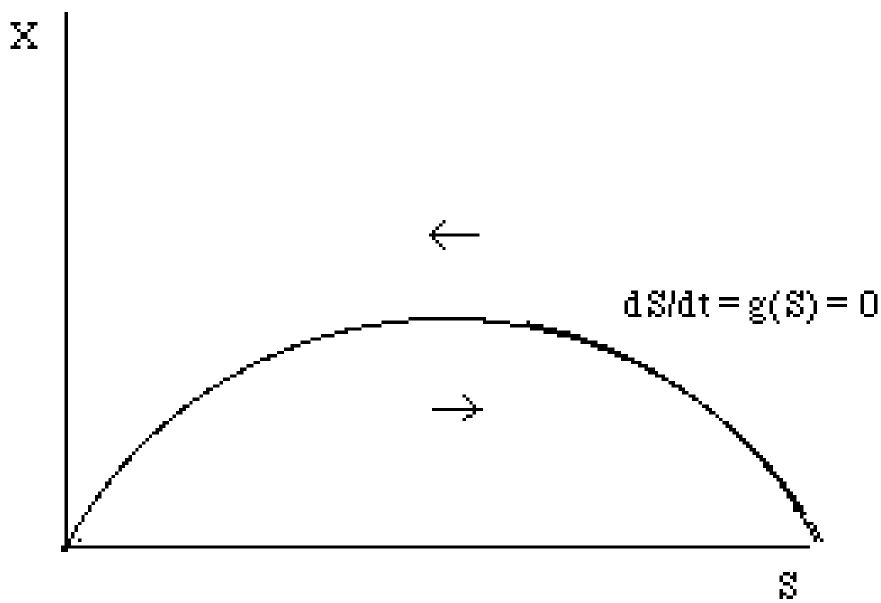
Furthermore, with optimal prices, the steady state equation for $\lambda_{t+1} - \lambda_t = 0$ can be expressed as a function of S and X by re-arranging equations (7) and (8).

$$[r - g_S(S)]\,[B(X) - c(S)] + Xc_S(S) = 0.$$

## Phase Plane Analysis

Phase plane analysis is a graphical method of analyzing the dynamic behavior of a bioeconomic system. It is useful for determining whether a bioeconomic system will be in steady-state, will drive the resource to extinction, or will cycle.

The curve below gives all the points at which "the biology is in steady state," i.e., where the stock will not rise or fall. This curve is simply the growth function g(S) that we have seen before. On the phase plane it is labeled "dS/dt = 0", i.e., the change in stock over time is zero.

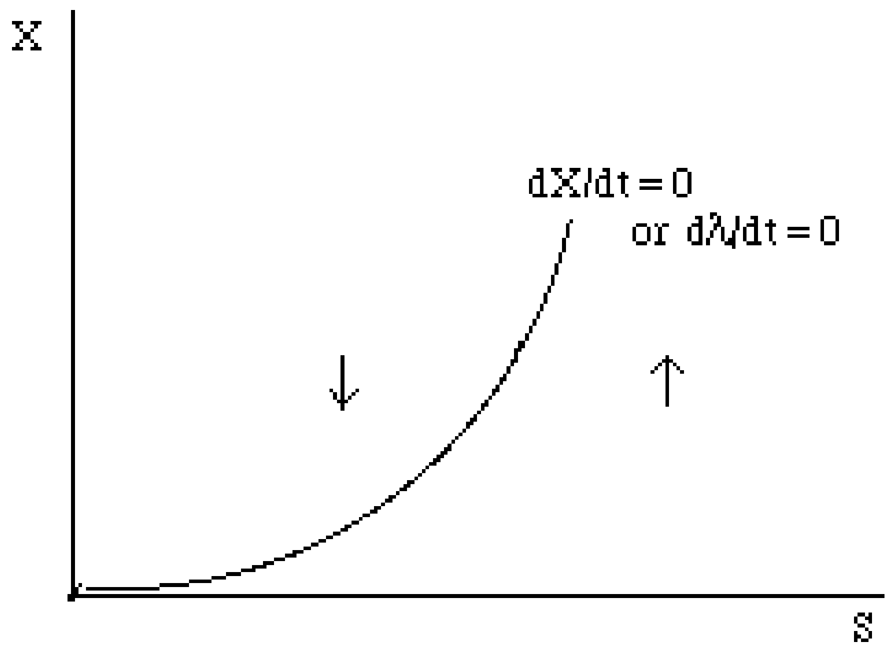



The arrows in the figure show that, if harvest is above the dS/dt = 0 line, i.e., if harvest is above steady-state harvest, then stock will fall, and if harvest is below the dS/dt = 0 line, stock will rise. **The line, dS/dt = 0, is commonly referred to as an iso-cline**.

# Phase Plane Analysis (cont.)

The second curve gives all the points at which "the economics is in steady state," i.e., where the economic agent has no incentive to either increase or decrease harvest X. The second curve is derived from the first order conditions when assuming the system is in economic steady state.

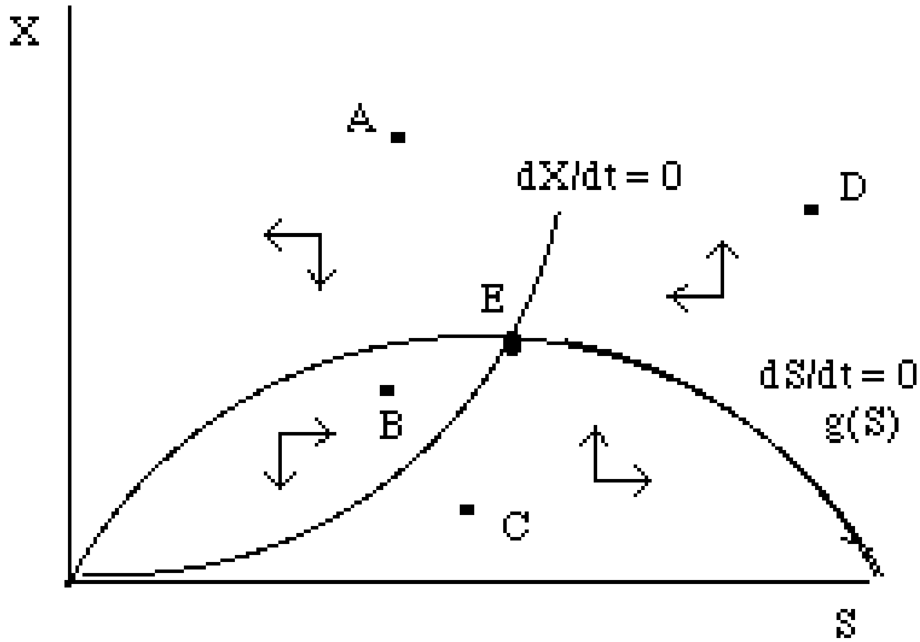
On the phase plane the second curve is either "dX/dt = 0," i.e., the change in harvest over time is zero, or "dλ/dt = 0," the change in (undiscounted) user cost over time is zero.

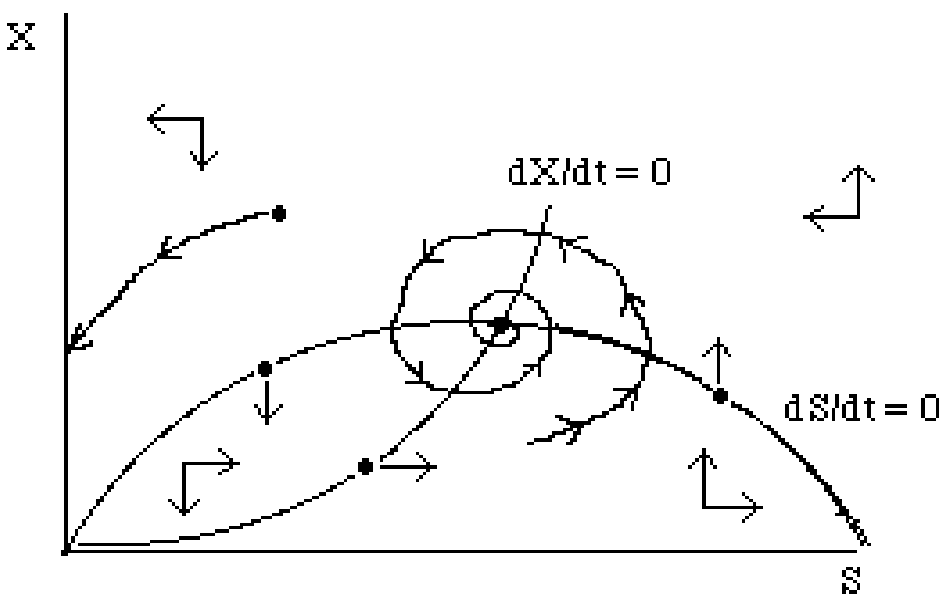



The arrows in the figure show that, if stock is to the right of the dX/dt = 0 line, i.e., if stock is high, fish are easy to find in the ocean and harvesting costs are therefore low, then harvest will rise, and if stock is to the left of the dX/dt = 0 line, harvest will fall in order to equate MB = MC of fishing.

# Phase Plane Analysis (cont.)

Putting the two steady-state curves together, we get:



# Phase Plane (cont.)

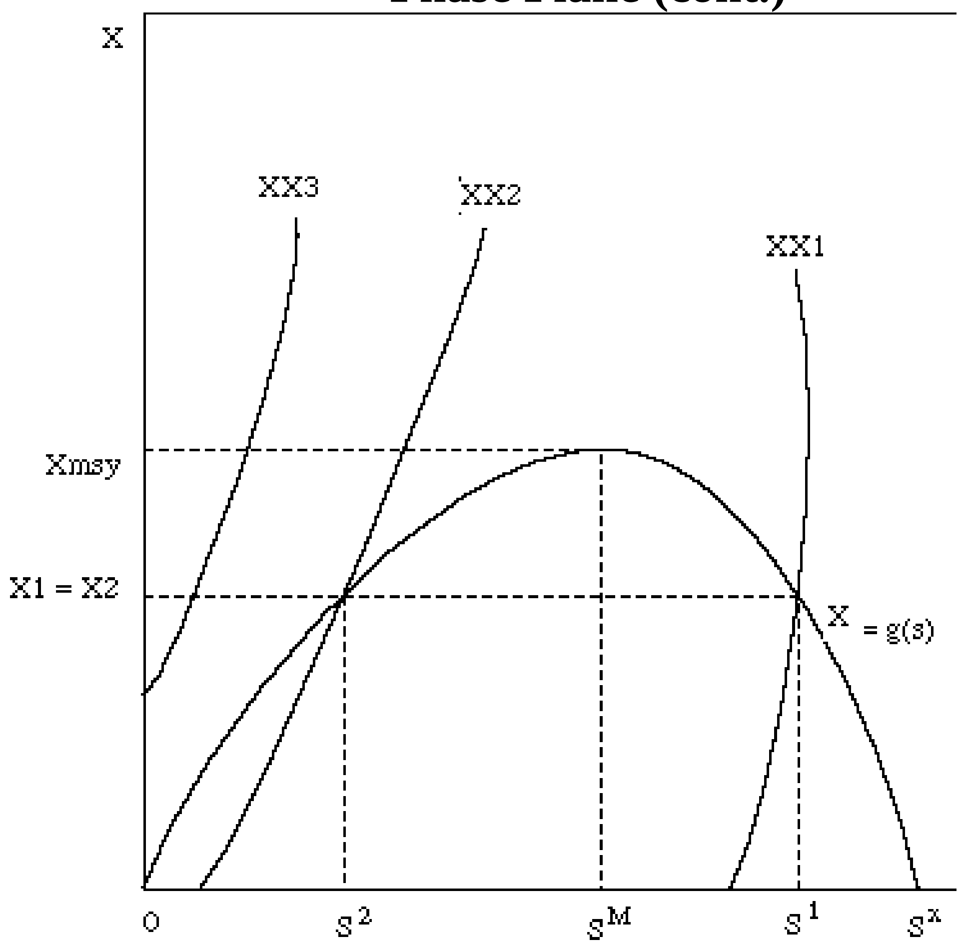



Figure 1: The possible scenarios of optimal management of renewable resources

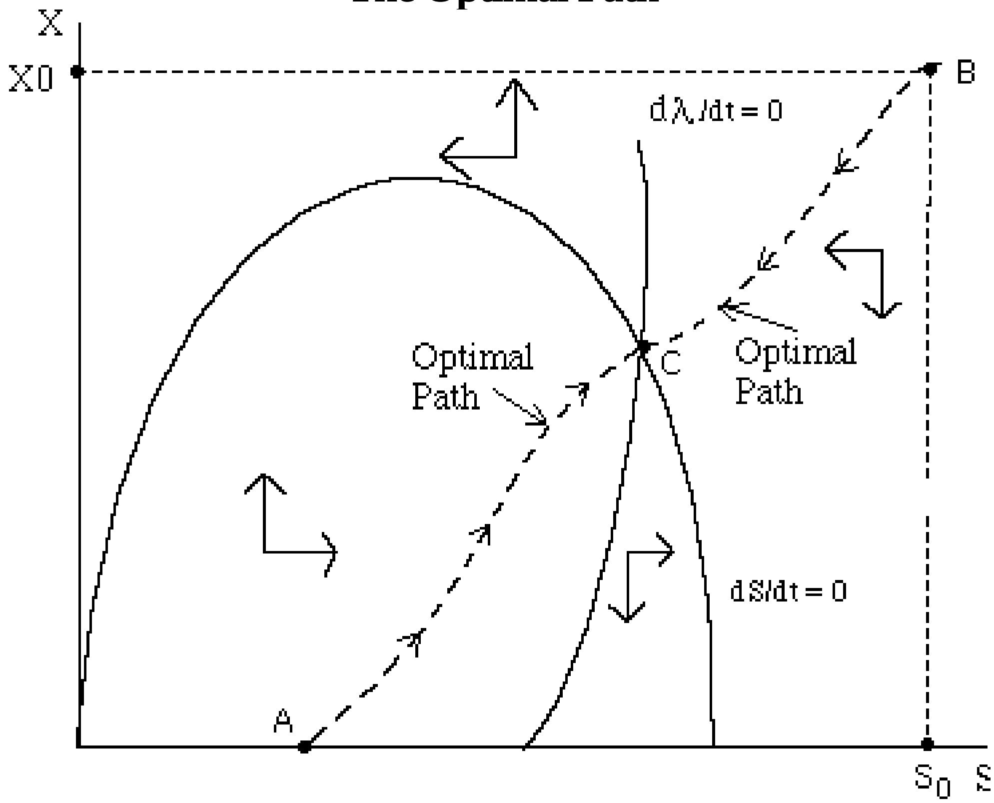
The curve, X = g(s), denotes all X, S combinations that lead to steady state of stock. The curves—$XX^1$, $XX^2$, and $XX^3$—denote all X, S combinations leading to steady state of stock *prices*.

- For each of these curves, $\left[r - g_S(S)\right]\left[B_X - C(S)\right] + XC_S(S) = 0$.

Each curve is drawn to represent a different interest rate:

- Curve $XX^3$ corresponds to the highest r,
- Curve $XX^1$ corresponds to the lowest r.

## The Optimal Path




Phase planes are often drawn with respect to shadow prices.

To derive North-South Arrows:

- Say the shadow price of fish jumps up in a given period to a point above the iso-cline $d\lambda/dt = 0$
  - -the harvest will increase because fish are worth more
- Say the shadow price of fish deviates downward in a given period to a point below the iso-cline $d\lambda/dt = 0$
  - -the harvest will decrease, because fish are undervalued

In this case, the optimal path is nearly a straight line (not a spiral). The optimal path gives the harvest level associated with any level of stock that will cause the system to converge to a steady-state optimal solution, at the equilibrium level C.