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A global game with strategic substitutes and complements

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Abstract

We study a global game in which actions are strategic complements over some region and strategic substitutes over another region. An agent's payoff depends on a market fundamental and the actions of other agents. If the degree of congestion is sufficiently large, agents' strategies are non-monotonic in their signal about the market fundamental. In this case, a signal that makes them believe that the market fundamental is more favorable for an action may make them less likely to take the action, because of the risk of overcrowding.

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"*Nobody goes there anymore; it's too crowded*"
—Yogi Berra

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1. Introduction

In many settings, the benefit to an agent of taking a particular action depends on the number of other agents who take the same action. In some cases, this relationship is non-monotonic. For example, positive network effects may make it more attractive for a firm to enter a new market if a few other firms also enter. However, if a large number of firms enter, the market becomes too crowded and further entry is unattractive. Agents may also have different information about a random variable that affects all of their payoffs, i.e., about a 'market fundamental'. For example, the market fundamental may be the true size of the market demand, or the true cost of production. The resulting incomplete information game is a coordination problem with congestion effects.

This binary action game contains two widely studied limiting cases: where the relative payoff is always decreasing (strategic substitutes) or always increasing (strategic complements) with the actions of others (see Bulow et al., 1985). In the former case, there is generically a unique equilibrium when the market fundamental is common knowledge. When actions are strategic complements, and there is common knowledge, there may be multiple equilibria. Important applications of the second model include Matsuyama (1991) and Krugman (1991) (two-sector models with increasing returns to scale that are external to the firm) Katz and Shapiro (1985) (network externalities) and Diamond and Dybvig (1983) (bank runs).

Carlsson and van Damme (1993) analyzed two-agent *global games*, in which the game selected by nature is observed by players with some noise. They show that, in the limit as the noise becomes small, iterated deletion of dominated strategies leads to a unique equilibrium. The key to the uniqueness result is that the support of any agent's higher order beliefs about other agents' signals becomes arbitrarily large for a sufficiently high order of beliefs. ("Higher order beliefs" are the beliefs that an agent has about other's beliefs about other's beliefs, and so on.) Further, the limit equilibrium is noise independent—it is the risk dominant equilibrium of the complete information game. As in Rubinstein (1989), iterated deletion eliminates equilibria that rely on expectations, leading to a unique equilibrium in which agents' strategies depend monotonically on their signals.

In games with a continuum of agents, Morris and Shin (1998) use this idea to analyze currency attacks, and Morris and Shin (2004) use it to analyze the pricing of debt.[2] In these papers, a small amount of incomplete information about the market fundamental leads to a unique equilibrium. This equilibrium is monotonic: in a binary action game, as the market fundamental becomes more favorable for a particular action, the fraction of agents who take that action increases (more exactly, never decreases).

We extend the previous analysis of global games by deriving qualitative properties of the set of symmetric equilibria in an incomplete information coordination game with regions of congestion. (Uniqueness is a more difficult issue, and is not addressed here.) In the case where the market fundamental is common knowledge, the equilibrium set in our game is easy to characterize. Over an intermediate range of values of the fundamental, there are multiple equilibria, each of which is monotonic in the fundamental. When there is not common knowledge, and the congestion effect is extremely weak, our model resembles the pure coordination games mentioned above.

[2] Other approaches to equilibrium selection have been developed. Herrendorf et al. (2000) show how heterogeneity in the manufacturing productivity (rather than the information) of agents in the Matsuyama model can remove indeterminacy and multiplicity of equilibrium. Frankel and Pauzner (2000), following Burdzy et al. (2001), show that exogenous shocks can lead to a unique equilibrium in the Matsuyama setting.

Not surprisingly, we find that in this case our model contains the kind of monotonic pure strategy equilibrium that the previous models identified as being unique.

Our result can be compared to the sufficient conditions for the existence of monotone pure strategy equilibria in incomplete information games in Athey (2001). Her main restriction is that a single-crossing condition be satisfied: the expected utility of each agent, when faced with rivals who are playing monotonic pure strategies, should satisfy the Milgrom and Shannon (1994) single-crossing property. She shows that games in which ex post payoffs are supermodular or log-supermodular in all players' actions and types, and in which types are affiliated, satisfy the single-crossing condition. Her model is quite general, allowing for interdependent valuations and general type distributions. In contrast, our model is more specific (as will be clear in Section 2). In return, we are able to provide a necessary and sufficient condition for existence of a monotone pure strategy equilibrium. The necessary and sufficient condition relates the degree of congestion (which should be weak) and the precision of agents' signals.

If the congestion effect is sufficiently strong, only non-monotonic equilibria exist. This result means that over some interval, as agents receive signals that suggest that the market fundamental is more favorable to a particular action, they become less likely to take that action. The explanation is that a more favorable signal informs the agent that others are also receiving more favorable signals, leading to the possibility that congestion will be severe. In some equilibria, there are disjoint intervals of signal space with the following characteristic: agents who receive signals in a particular interval take the action with probability 1, whereas agents who receive signals that reflect a better market fundamental refrain from the action with probability 1.

Although economic situations motivate our model, it is applicable to more general social situations, and possibly to natural environments.[3] For example, agents' utility from going to a bar may depend on the quality of the music (the market fundamental) and the size of the crowd.[4] They prefer the bar when it is neither empty nor extremely crowded. Yogi Berra observed that "Nobody goes there anymore; it's too crowded". He may have exaggerated, but he had the right idea.

2. The model

An individual is considering which of two actions to take, $a \in \{0, 1\}$. To make the situation concrete, we speak of a visit to a bar and associate the action $a = 0$ with "don't go to the bar" and the action $a = 1$ with "go to the bar". The individual is small and therefore behaves non-strategically, and the mass of individuals is 1. The utility that each individual receives from choosing 'go' depends on two factors: an underlying state, denoted $\theta \in \mathbf{R}$; and the fraction of individuals who undertake the action, $\alpha \in [0, 1]$. The payoff function $U(\theta, \alpha)$ from choosing 'go' satisfies:

Assumption 1. $U(\theta, \alpha) = \theta + f(\alpha)$, where $f(\alpha) : [0, 1] \to \mathbf{R}$.

Hence the payoff from going to the bar increases with the state: $\partial U/\partial\theta > 0$. Additive separability ($\partial^2 U/\partial\theta\, \partial\alpha = 0$) does not affect qualitative aspects of the main results but simplifies

[3] As an example of a natural environment, consider a herd that is grazing in a particular location. Individuals can move to another, possibly superior, location. If only a small number of individuals move, they become exposed to predators, but if many move the new field becomes too crowded.

[4] Arthur's (1994) bar analogy is based on a common knowledge game with strict substitutes (pure congestion effects).

the analysis. The individual receives zero utility from not going to the bar. In discussing pure strategies, we assume that an individual who is indifferent between the two actions does not go to the bar; this assumption plays no substantial role.

The interaction term $f(\alpha)$ satisfies:

Assumption 2. (i) $f(\alpha)$ is an analytic function (and hence continuously differentiable in α and bounded over [0, 1]); (ii) $f(0) = 0$; (iii) $f'(0) > 0$; (iv) $f(\alpha)$ is quasi-concave.

Part (i) of the assumption could be generalized; for the purpose of this paper, the generalization would not be interesting. Part (ii) is simply a normalization. Part (iii) ensures that there is a region over which individuals' actions are strategic complements: that is, the payoff to each individual of choosing 'go' increases with the proportion of others who choose 'go', when that proportion is sufficiently small. Part (iv) ensures that the interaction function is uni-modal; this assumption could be relaxed at the cost of additional notation, but would not produce additional insights.

Our main objective is to analyze the equilibrium set when $f(\cdot)$ is non-monotonic and there is incomplete information about θ. Nature chooses the state according to a uniform distribution on the real line. (Morris and Shin, 2003 provide a justification for the assumption of improper, or diffuse, priors.) Each individual i receives a private signal $x_i = \theta + \eta_i$ where η_i is a random variable drawn uniformly from the interval $[-\epsilon, \epsilon]$ with $\epsilon > 0$. Conditional on θ, the signals are independently drawn from an identical distribution across individuals. The distributional assumptions imply that the agent's posterior distribution is uniform. (Carlsson and van Damme, 1993 and Burdzy et al., 2001 use more general distributional assumptions. The assumption of a uniform distribution simplifies the calculations in our proofs. We do not think it is critical for our results, although we do not have any formal proofs for this conjecture.)

The extensive form is that individuals choose actions simultaneously, after observing their private signal. The structure of the game is common knowledge. Before receiving private information (if any), agents are identical. Since agents are ex ante identical and strategically negligible, we concentrate on symmetric equilibria.[5]

We use the following definitions:

Definition 1. (i) $\overline{f} \equiv \max_{\alpha\in[0,1]} f(\alpha)$; (ii) $\underline{\theta} \equiv -\overline{f}$; (iii) $\bar{\theta} \equiv -\min[0, f(1)]$; (iv) $\hat{\theta} \equiv -f(1)$.

Assumption 2 and Definition 1 imply the following relations: (i) $\overline{f} > 0$, (ii) $\underline{\theta} < 0$, (iii) $\bar{\theta} > \underline{\theta}$; (iv) $\underline{\theta} \leqslant \hat{\theta} \leqslant \bar{\theta}$. We say that there is *weak congestion* when $f'(\alpha) < 0$ for some α and $f(1) \geqslant 0$. In this case, for $\theta \geqslant 0$ an agent's utility remains positive, even though a more crowded bar decreases utility. There is *strong congestion* when $f(1) < 0$.

[5] In an asymmetric equilibrium, two agents receiving the same private information would choose different actions. But these two agents have the same signal and face the same expected action profile of their opponents. Hence the expected payoffs of the two agents will be the same. We then assume that the two agents take the same action. This is without loss of generality when the agents are not indifferent between the two actions. When the agents are indifferent, they could choose different actions, but we rule out this possibility. We regard it as less interesting, because in a mixed strategy equilibrium in games with a continuum of agents, a symmetric mixed strategy equilibrium is observationally equivalent to uncountably many asymmetric mixed strategies. (If the symmetric mixed strategy requires agents to take an action with probability p, an observationally equivalent asymmetric mixed strategy can require the mass q to take that action with probability $\tilde{p} = p/q$.)

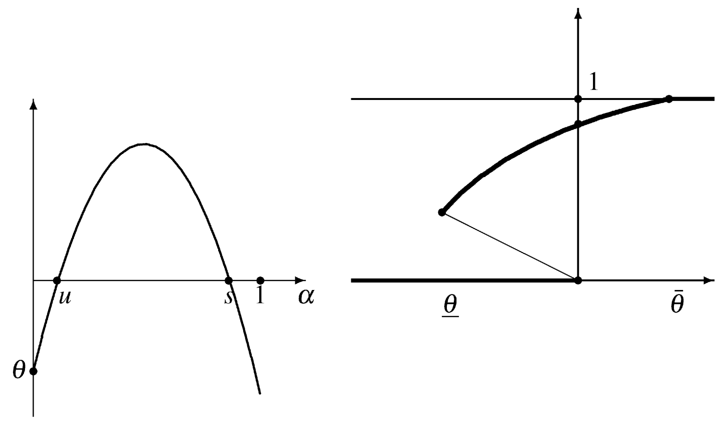

Fig. 1. The payoff function (a) and the equilibrium correspondence (b).

We start by considering the case in which θ is common knowledge. The values $\underline{\theta}$ and $\bar{\theta}$ define *dominance regions*: for $\theta < \underline{\theta}$, it is a strictly dominant action for each individual to choose 'don't go', while for $\theta > \bar{\theta}$, it is a strictly dominant action for each individual to choose 'go'. For $\theta \in [\underline{\theta}, \bar{\theta}]$, there is a coordination problem: there is no strictly dominant action for any individual, and the optimal action depends on the proportion of individuals taking the action. For $\theta > \hat{\theta}$, it is an equilibrium for all individuals to choose 'go' when θ is common knowledge.

Figure 1 illustrates the model in the case where θ is common knowledge and $f(\alpha)$ is non-monotonic. Panel (a) graphs the utility function for a particular $\theta < 0$ (the vertical intercept) outside the dominance region. For this value of θ there is a pure strategy equilibrium 'don't go', and two mixed strategy equilibria indicated by points s and u. The former is stable, in the sense that any perturbation to the proportion of agents choosing 'go' is self-correcting. (If, for some reason, a greater proportion than s of agents chooses 'go', then those agents who now choose 'go' receive negative expected utility and so will change their action.) By the same reasoning, the point u is unstable. Panel (b) graphs the equilibrium α correspondence for different values of the publicly observed parameter θ. In this example $\bar{\theta} > 0$, so there is strong congestion. For $\theta \leqslant \underline{\theta}$, the unique pure strategy equilibrium is 'don't go' (i.e., $\alpha = 0$); for $\underline{\theta} < \theta < 0$, there is an unstable equilibrium (the light downward sloping curve) and two stable equilibria, indicated by the thick lines: a mixed strategy and the pure equilibrium 'don't go'. For $0 \leqslant \theta$, there is a unique equilibrium, either a stable mixed strategy or the pure strategy 'go', depending on the magnitude of θ. The two mixed strategy equilibria are strongly monotonic in θ and the pure strategy equilibria are weakly monotonic in θ.

This model contains the special case where $f(\cdot)$ is everywhere increasing—i.e., actions are strategic complements for all α. In this situation there are two pure strategy equilibria for θ outside the dominance regions, and the unstable mixed strategy equilibrium. In the absence of Assumption 2(iii), the model also contains the case where $f(\cdot)$ is everywhere decreasing—i.e., actions are strategic substitutes for all α. In this case there is a unique equilibrium, which is either pure or mixed depending on θ.

In the case of incomplete information, a strategy for player i is a probability measure on the product of the signal space and the action space; that is, players use distributional strategies. (See below for a formal definition.) A Bayesian Nash equilibrium is a profile of strategies (one for each agent) such that the agent's strategy maximizes its expected payoff conditional on the information available, when all other individuals follow the strategies in the profile. We consider only symmetric equilibria: agents play a different strategy only because they have different signals. We consider four classes of equilibria, depending on whether the equilibrium is pure or mixed, and monotonic or non-monotonic.

3. Equilibrium with incomplete information

Previous analysis of global games shows that when there is incomplete information and global complementarities (so that for all action choices, actions are strategic complements), there is a unique equilibrium in monotone pure strategies (which we call 'switching' strategies: see below for a formal definition). We show that this kind of equilibrium exists (for all ϵ) in our setting if and only if the degree of non-monotonicity of the interaction function $f(\alpha)$ is sufficiently small—that is, if the amount of congestion is small. When this condition is not satisfied, equilibria are not monotonic in the signal.

Although we cannot prove the existence of *non-monotonic* pure strategy equilibria, we show that if they exist, then for small ϵ the intervals of signal space over which it is optimal to take one action alternate frequently with intervals for which the other action is optimal. Thus, for small ϵ and sufficiently large congestion, any pure strategy equilibrium gives rise to what appears to be erratic behavior. Next we also show that there can be no monotonic equilibrium in mixed strategies.

As explained in the next subsection, we know that an equilibrium exists for this game. Consequently, the results described above imply that if the amount of congestion is sufficiently great, the equilibrium decision rule must be non-monotonic in the signal.

3.1. Existence of equilibrium

In the incomplete information game, we suppose that players use *distributional strategies*, as defined by Milgrom and Weber (1985). A distributional strategy for player i, conditional on the state θ, is a probability measure y_i on the subsets of $[\theta - \epsilon, \theta + \epsilon] \times \{0, 1\}$ (i.e., the Cartesian product of the player's type and action space), for which the marginal distribution on the set of types is uniform on $[\theta - \epsilon, \theta + \epsilon]$. Since the payoff to a player from choosing action 0 is 0, and since a player affects others' payoffs only if it chooses action 1, we can concentrate on the probability measure $y_i : [\theta - \epsilon, \theta + \epsilon] \times \{1\} \to [0, 1]$. From this point on, we refer to this measure with the short-hand $y_i(x_i)$.

One potential problem in trying to apply Milgrom and Weber's existence theorem 1 in our game arises from the dependence of types, due to the correlation induced by the state variable θ. This feature means that Milgrom and Weber's 'absolutely continuous information' condition is violated in our game; and hence that expected payoffs may not be continuous (in a topology that ensures compactness of strategy sets). Consequently, it may not be possible to apply a standard fixed point theorem (such as Glicksberg's).

Fortunately, particular features of our game allow us to overcome this problem. First, as we noted above, we can concentrate on symmetric equilibria. We can therefore replace player i's opponents with a 'representative' opponent (for any $i \in [0, 1]$). This means that the game can be treated as if there are two players, rather than a continuum, without loss of generality in determining the equilibrium. This allows us to avoid the technical issues that arise with an uncountable infinity of players (for example, the restriction to a finite state space: see, e.g., Kim and Yannelis, 1997).

Secondly, the particular form of the expected payoff means that establishing continuity is relatively straightforward. The proportion of individuals taking the action $a = 1$, given the symmetric strategy profile $\{y(x)\}$ and a realization of the state θ, is

$$\alpha(\theta, y) = \frac{1}{2\epsilon} \int_{\theta-\epsilon}^{\theta+\epsilon} y(z)\,\mathrm{d}z.$$

The next lemma is an immediate consequence of the continuous and conditionally independent distributions of individuals' signals, and so is stated without proof.

Lemma 1. *$\alpha(\theta, y)$ is a continuous in θ for any strategy profile $\{y(\cdot)\}$.*

When the strategy profile $\{y(\cdot)\}$ is played, player i who receives a private signal of x and plays the strategy $y_i(\cdot)$ receives an expected payoff of

$$u(x, y_i, y) = \left(\frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} \bigl(\theta + f\bigl(\alpha(\theta, y)\bigr)\bigr)\,\mathrm{d}\theta\right) y_i(x).$$

The following proposition establishes existence of equilibrium in our game. (All proofs are contained in Appendix A.)

Proposition 1. *In the game of incomplete information, there is an equilibrium in distributional strategies.*

3.2. Monotonic pure strategy equilibrium

We start the characterization of equilibrium by considering the switching—i.e., monotonic pure strategy—equilibrium identified in the global game papers:

$$S_k(x_i) = \begin{cases} \text{go} & \text{if } x_i > k, \\ \text{don't go} & \text{if } x_i \leqslant k. \end{cases}$$

Given the threshold $k = x^*$,

$$\alpha(\theta, S_{x^*}) = \mathbb{P}(\theta + \eta \geqslant x^*) = \begin{cases} 0 & \theta < x^* - \epsilon, \\ \frac{\theta - x^* + \epsilon}{2\epsilon} & x^* - \epsilon \leqslant \theta \leqslant x^* + \epsilon, \\ 1 & \theta > x^* + \epsilon. \end{cases} \tag{1}$$

If a monotonic switching equilibrium is played, then an individual who receives a signal equal to the equilibrium switch point, x^*, is indifferent between playing 'go' and 'don't go'. The fact that utility is additively separable in θ and α and that the conditional expectation of θ is monotonic in the signal means that there is a unique value of x^* that ensures indifference. Lemma 2 determines this unique x^*.

Lemma 2. *The expected payoff $u(x^*, S_{x^*})$ of an agent with a signal equal to the switch point, x^*, is zero if and only if $x^* = -\int_0^1 f(\alpha)\,\mathrm{d}\alpha$.*

Our chief result in this section is that the existence of a switching equilibrium requires a combination of an imprecise signal (large ϵ) and a small amount of congestion. Therefore, as $\epsilon \to 0$, the existence of a switching equilibrium requires that the amount of congestion be small. The intuition for this result is apparent from the middle part of Eq. (1). For a signal equal to x^*, agents are indifferent between the two actions. For values of θ in the neighborhood of x^*, $\frac{\partial\alpha}{\partial\theta} = \frac{1}{2\epsilon}$. A small value of ϵ means that α is very sensitive to changes in θ, and that a signal provides precise information about the value of θ. Agents who receive signals in the neighborhood of x^* have very different beliefs about the value of α but similar beliefs about the value of θ, when ϵ is small.

Compare the beliefs of two agents who, respectively, receive signals $x^* + \delta$ and $x^* + 2\epsilon - \delta$; δ is a positive number, small relative to ϵ. As $\delta \to 0$, the first agent is indifferent between the two actions. The second agent is nearly certain that $\alpha = 1$. If there is significant congestion, the first agent expects a much higher value of the interaction component ($f(\alpha(x^*)) > f(1)$), but the two agents' expectation of θ differs only by 2ϵ. Therefore, when congestion is large and ϵ is small, the first agent's expected payoff from going to the bar is higher than the second agent's. Since the first agent is indifferent, the second agent must prefer not to go to the bar. Consequently, when ϵ is small and congestion is significant, a switching equilibrium does not exist.

We now provide a formal statement of this result. The following proposition provides the necessary and sufficient conditions for the existence of a switching equilibrium.

Proposition 2. *The switching strategy around* $x^* = -\int_0^1 f(\alpha)\,\mathrm{d}\alpha$ *is an equilibrium if and only if*

$$2\epsilon \geqslant \max\left\{\max_{z\in(-1,0)}\left(\frac{1}{z}\int_{z+1}^{1} f(\alpha)\,\mathrm{d}\alpha\right), \max_{z\in(0,1)}\left(\frac{1}{z}\int_{0}^{z} f(\alpha)\,\mathrm{d}\alpha - f(1)\right)\right\}. \tag{2}$$

The following two remarks are less general but more intuitive than Proposition 2. Remark 1 shows that the sufficient condition is satisfied if the amount of congestion is small enough. Remark 2 shows that the necessary condition fails if the amount of congestion is large enough. Recall that $\overline{f}$ is defined as the maximum value of $f(\alpha)$. If Assumption 2(iv) holds, then $\overline{f} - f(1)$ is one measure of the amount of congestion, and of the importance to an agent of making correct inferences about how other agents will behave.

Remark 1. A sufficient condition for an equilibrium in switching strategies to exist is $\overline{f} - f(1) \leqslant 2\epsilon$.

Remark 2. If $f(1) < -x^*$, then there is an $\bar{\epsilon} > 0$ such that when $\epsilon < \bar{\epsilon}$, the switching strategy around x^* is not an equilibrium.

The sufficient condition in Remark 1 limits the amount of congestion. The maximum amount of congestion (consistent with the existence of an equilibrium switching strategy) decreases with the precision of the private signals; for arbitrarily (but not perfectly) precise signals, the interaction function $f(\cdot)$ must be non-decreasing. In fact, the condition in Remark 1 determines when an 'iterated deletion of strictly dominated strategies' argument can be used to derive the switching equilibrium of Proposition 2. To see why, recall that the iterated deletion argument used in Morris and Shin (2003) relies on two facts about the expected payoff $u(x, S_k)$: that it is increasing in x and decreasing in k. These facts hold in a global game in which actions are strategic complements. In a game with congestion, however, they need not hold. In particular, the proof of Remark 1 shows that $u(x, S_k)$ is increasing in x for all k if and only if $\overline{f} - f(1) \leqslant 2\epsilon$. When this condition is violated, the iterated deletion argument cannot be applied to determine a unique equilibrium.[6] And in fact, iterated deletion makes very little progress when the condition in Remark 1 does not hold. To see this, suppose that $f(1) < 0$ (i.e., there is strong congestion) and $2\epsilon > -f(1)$; and consider iterated deletion from the right-hand dominance region, defined by $x \geqslant \bar{\theta} \equiv -f(1) > 0$. It is then easy to verify that

$$u(x, S_{\bar{\theta}}) = \begin{cases} x & x < \bar{\theta} - 2\epsilon, \\ x + \frac{1}{2\epsilon}\int_{\bar{\theta}-\epsilon}^{x+\epsilon} f\left(\frac{\theta - (\bar{\theta}-\epsilon)}{2\epsilon}\right) \mathrm{d}\theta & \bar{\theta} - 2\epsilon \leqslant x < \bar{\theta}, \\ x + \frac{1}{2\epsilon}\int_{x-\epsilon}^{\bar{\theta}+\epsilon} f\left(\frac{\theta - (\bar{\theta}-\epsilon)}{2\epsilon}\right) \mathrm{d}\theta + (x - \bar{\theta}) f(1) & \bar{\theta} \leqslant x < \bar{\theta} + 2\epsilon, \\ x + f(1) & x \geqslant \bar{\theta} + 2\epsilon. \end{cases} \tag{3}$$

For the iterated deletion argument to work, we need to show that there is a unique value $\hat{x} < \bar{\theta}$ such that $u(\hat{x}, S_{\bar{\theta}}) = 0$. From Eq. (3), $u(x, S_{\bar{\theta}}) > 0$ for all $x \geqslant \bar{\theta}$; and $u(\bar{\theta} - 2\epsilon, S_{\bar{\theta}}) > 0$ and $u(x, S_{\bar{\theta}})$ is strictly decreasing in x for $x < \bar{\theta} - 2\epsilon$, so that there is a single point in the interval $(-\infty, \bar{\theta} - 2\epsilon]$, given by $x = 0$, at which $u(x, S_{\bar{\theta}}) = 0$. We cannot determine, however, what happens to $u(x, S_{\bar{\theta}})$ in the interval $x \in (\bar{\theta} - 2\epsilon, \bar{\theta})$: it need not be increasing in x, and it need not be positive throughout the entire interval.[7] In short, iterated deletion breaks down at the first step.

The sufficient condition in Remark 2 is the converse of the condition in Remark 1. When there is enough congestion and signals are sufficiently precise, then the monotonic switching strategy cannot be an equilibrium. The monotonicity of a switching strategy would make more agents go to the bar for high signals. On receiving a high signal, an agent knows that many others have received a high signal (since signals are precise); therefore the bar will be crowded. With congestion, the agent with a high signal prefers not to go to the bar.

We now consider two examples where we can relate the existence of a switching equilibrium to the model's parameter values. In the first example, the amount of congestion is independent of the uncertainty parameter ϵ. For this example, a switching equilibrium exists for all $\epsilon > 0$ only if there is weak congestion. In the second example, the amount of congestion depends on the uncertainty parameter ϵ. In this case, a switching equilibrium can exist even if there is strong congestion.

Example 1. Let $f(\alpha) = \alpha - b\alpha^2$, $b \geqslant 0$. In this case we can compute the integrals in Eq. (2). Since ϵ can be arbitrarily close to 0, Proposition 2 implies that a switching equilibrium exists for all $\epsilon \geqslant 0$ if and only if

$$0 \geqslant \max\left\{\max_{z\in[-1,0)}\left(\frac{1}{3}bz^2 + \left(b - \frac{1}{2}\right)z + b - 1\right), \max_{z\in[0,1]}\left(\frac{1}{2}z - \frac{1}{3}bz^2\right)\right\}.$$

It is straightforward to show that this inequality holds if and only if $b \leqslant 0.75$.

For this example, $f'(1) = 1 - 2b$. Thus, a switching equilibrium exists even if there is congestion ($f'(1) < 0$) provided that the amount of congestion is small ($f(1) = 1 - b \geqslant 0.25 > 0$). In this case, there is weak congestion, since $f(1) > 0$.

[6] Note that the iterated deletion argument does not necessarily have to apply to prove the existence of a switching equilibrium. That is why the condition in Remark 1 is only *sufficient* for existence.

[7] Of course, by continuity, it must be positive in some neighborhood of both $\bar{\theta} - 2\epsilon$ and $\bar{\theta}$.

Example 2. Let $f(\alpha) = (\alpha - b\alpha^2)\epsilon$, $b \geqslant 0$, $\epsilon \geqslant 0$. As with Example 1, we can compute the integrals in inequality (2). In this case, however, we can cancel ϵ from both sides, so that the inequality is independent of ϵ. After some simplification, we can rewrite inequality (2) as

$$0 \geqslant \max\Big\{\max_{z\in(-1,0)}\big(2bz^2 + (6b-3)z + 6b - 18\big), \max_{z\in(0,1)}\big(-2bz^2 + 3z + 6b - 18\big)\Big\}.$$

Some straightforward calculation shows that this inequality is satisfied if and only if $b < 3.0$.

For both Examples 1 and 2, the sufficient condition in Remark 1 holds for all $\epsilon \geqslant 0$ only if $b \leqslant 0.5$. In this circumstance there is no congestion: $f'(\alpha) \geqslant 0$ for all α. Thus, the examples show the extent to which the sufficient condition in Remark 1 is too strong.

The results above reveal the role of signals as a coordination device in the class of games under investigation. An equilibrium determines the distribution of strategies used by other agents conditional on the private signal, since a strategy is a function of a signal whose conditional distribution is known. When the noise decreases, the distribution of signals becomes concentrated. This concentration allows agents to predict the equilibrium effect of other agents' strategies with greater precision, relative to the fundamental.

The fundamental θ has a direct effect on the payoff and an indirect (or strategic) effect via the equilibrium value of α. A decrease in the noise has no effect on $\mathbb{E}(\theta)$ and therefore does not alter the direct effect of the fundamental. However, using Eq. (1), for interior values of α and for a given threshold strategy, $\frac{\partial\alpha}{\partial\theta} = \frac{1}{2\epsilon}$; the strategic effect $(\frac{\partial U}{\partial\alpha}\frac{\partial\alpha}{\partial\theta})$ of the fundamental increases as the noise decreases.

The agent's expectation of θ always increases linearly with the signal, but in the region slightly above the threshold, a larger signal has a large strategic effect on the agent's expected payoff. If the distance between the threshold and the upper dominance region is large relative to ϵ, the agent expects that all other agents will go to the bar even when the signal indicates that the fundamental is not good enough to justify going under this circumstance. Hence, smaller noise tends to eliminate the monotonic pure strategy equilibrium.

Thus, with congestion, an equilibrium of the game may not be monotonic. Since monotonicity is a crucial property in establishing uniqueness in global games without congestion, we conjecture that the equilibria may not be unique when there is congestion.

3.3. *Non-monotonic pure strategy equilibrium*

When the condition given by Proposition 2 is not satisfied—i.e., if the congestion effect is sufficiently large—a switching equilibrium does not exist. In this case, any symmetric pure strategy equilibrium is non-monotonic. In this subsection, we provide a characterization of non-monotonic pure strategy equilibria under the (unproven) assumption that they exist.

In the non-dominance region, there is at least one interval such that an agent who receives a signal in the interval(s) chooses 'go' with probability 1, and an agent who receives a signal outside the interval(s) chooses 'don't go' with probability 1. We refer to these intervals of signals where agents choose 'go' with probability 1 as *islands*. As before, there are also the upper and lower dominance regions.

The 'lowest island' is the island with the smallest lower boundary. For all signals below the lowest island, agents choose 'don't go'. The distance between the upper boundary of this island and the nearest island (or the boundary of upper dominance region in the case where there is only one island) is finite. If that distance is greater than 2ϵ, we define the lowest island as *isolated*.

Similarly, any other island is isolated if its boundaries are greater than 2ϵ from the boundary of the nearest island or of the upper dominance region. The following proposition states that the islands cannot be very large, and they cannot be very far apart.

Proposition 3. *In any pure strategy non-monotonic equilibrium, none of these islands is isolated. If $f(1) < 0$, then none of the islands has length greater than 4ϵ. If $f(1) > 0$, there are never adjacent 'go' intervals (i.e., islands or the half-line that includes the upper dominance region, where 'go' is the equilibrium action) that are both longer than 4ϵ.*

This proposition implies that even if a pure strategy equilibrium exists when there is significant congestion, the equilibrium behavior changes rapidly with the signal. In that sense, the behavior appears erratic.

3.4. Mixed strategy equilibrium

In describing the game when θ is common knowledge, we pointed out that for some range of θ there is a stable and an unstable mixed strategy equilibrium. From Fig. 1, it is obvious that in this game the mixed strategy equilibria are monotonic in θ. This observation might suggest that even in cases where a monotonic pure strategy (i.e., switching) equilibrium does not exist in the game without common knowledge, there might be a monotonic mixed strategy equilibrium. This subsection shows that conjecture to be false. There never exists a monotonic mixed strategy equilibrium, regardless of whether there is a monotonic pure strategy equilibrium.

Consider a mixed strategy $y_M(\cdot)$ that is monotonic. Since the strategy is monotonic, it has countably many points of discontinuity in its domain; see e.g., Kolmogorov and Fomin (1970, Theorem 3, p. 316). Therefore $\alpha(\theta; y_M)$ is continuous (in θ), since it is the Lebesgue integral of $y_M(\cdot)$. Hence the expected payoff

$$x + \frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} f\big(\alpha(\theta, y_M)\big)\,\mathrm{d}\theta$$

is differentiable (in x), since it too involves a Lebesgue integral.

In a mixed strategy equilibrium, an agent who observes a signal x in the interval for which randomization takes place receives zero expected payoff:

$$x + \frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} f\big(\alpha(\theta, y_M)\big)\,\mathrm{d}\theta = 0. \tag{4}$$

Since the expected payoff on the left-hand side of Eq. (4) is differentiable a.e., by the argument above, and since the agent must receive zero expected payoff at all signals for which randomization takes place, the derivative of the expected payoff must also be zero in the mixing interval:

$$\frac{\partial}{\partial x}\left(x + \frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} f\big(\alpha(\theta, y_M)\big)\,\mathrm{d}\theta\right) = 0. \tag{5}$$

This equation implies

$$f\big(\alpha(x+\epsilon, y_M)\big) - f\big(\alpha(x-\epsilon, y_M)\big) = -2\epsilon. \tag{6}$$

The mixed strategy equilibrium must satisfy Eqs. (4) and (6).

If there were two or more disjoint randomization intervals, the equilibrium strategy would necessarily be non-monotonic. Since we are interested in monotonic mixed strategy equilibria, we can therefore restrict attention to mixed strategy equilibria in which there is a single randomization interval, $[\underline{x}, \bar{x}]$ say. For any signal below that interval, agents choose 'don't go' with probability 1, and for any signal above that interval they chose 'go' with probability 1:

$$y_M(x) \begin{cases} = 0 & x < \underline{x}, \\ \in [0,1] & x \in [\underline{x}, \bar{x}], \\ = 1 & x > \bar{x}. \end{cases} \tag{7}$$

In such an equilibrium, any individual who receives a private signal in this interval must be indifferent between choosing 'don't go' and 'go'; i.e., the conditional expected payoff function satisfies Eqs. (4) and (6).

We now show that this type of strategy cannot occur in equilibrium.

Proposition 4. *Any equilibrium strategy with a single randomization interval is non-monotonic.*

In summary, we have shown that *any* mixed strategy equilibrium must be non-monotonic. Either it involves two or more randomization intervals, in which case non-monotonicity is obvious. Or it involves one randomization interval, in which case Proposition 4 establishes non-monotonicity.

Proposition 4, together with the necessary and sufficient condition for monotonic pure strategy equilibrium in Proposition 2, imply that, with sufficient congestion, *any* equilibrium of the incomplete information game must be non-monotonic. This result is in contrast to the complete information game, where all equilibria are monotonic. The proofs of the propositions make clear how this difference arises. For intermediate signals, the expected disutility from congestion outweighs any direct benefit from going to the bar. Hence, a pure strategy switching equilibrium cannot exist. As for a mixed strategy equilibrium: the indifference condition that must hold in equilibrium means that, as an agent's signal about the bar goes up, the expectation of the interaction term must go down (to keep the agent indifferent). We show (in the proof of Proposition 4) that when the interaction function $f(\cdot)$ is non-monotonic (i.e., there is congestion), then if the strategy were monotonic in the signal, the equilibrium proportion of agents going to the bar would be non-monotonic in the fundamental. These two characteristics cannot both be true. Hence any mixed strategy equilibrium must be non-monotonic.

3.5. Numerical examples

When the condition in Proposition 2 is not satisfied we know that any equilibrium is non-monotonic, but we do not know whether the equilibrium is unique, or whether it consists of pure strategy islands or is a mixed strategy. Numerical methods provide some insight into these questions.

We use a finite-state approximation to the continuous state model. Rather than being an element of the real line, here θ can take one of N possible values with equal probability, $N < \infty$. Computing constraints place limits on the magnitude of N. When $\theta = \theta_i$, the signal can equal θ_i or any of the M values above and below θ_i with equal probability. The grid points are evenly spaced and we retain the assumption of uniformity. Since the support of θ, conditional on the signal θ_i, contains $2M + 1$ elements, and the prior support contains N elements, $\frac{2M+1}{N}$ is an inverse measure of the degree to which the signal is informative, as is ϵ in the continuous state space model. For a given N, a smaller value of M in this model corresponds to a small value of ϵ in the continuous model. In view of Proposition 2, the interesting case is where ϵ is small, so we emphasize $M = 1$ (although we also ran simulations for other values). We report results for values of N ranging from 50 to 60.

We use the quadratic formulation $f(\alpha) = \alpha(1 - b\alpha)$ from Example 1 of Section 3.2. We are interested in the case where there is congestion, so we considered only $b > 0.5$. Appendix A describes the numerical methods. Our results are as follows:

(1) For $b \leqslant 0.75$ the only equilibrium we find is the switching equilibrium. For b slightly greater than 0.75 the only equilibrium we find consists of a single pure-strategy island. For values of b larger than 0.83 we find no pure strategy equilibria.
(2) For b slightly smaller than 0.83 we find mixed strategy equilibria in addition to the pure strategy equilibrium. For larger values of b we find only mixed strategy equilibria. These are not unique.
(3) All of the mixed strategy equilibria are non-monotonic in the signal. In some case they consist of a set of signals for which the probability of going is positive, followed by a set of signals where the probability is 0 ("mixed strategy islands"); in other cases, although the probabilities are non-monotonic there are no 0-probability signals between positive-probability signals.

Since we use a finite state space model here, and since computing constraints limit the magnitude of N, we cannot guarantee that the numerical results also apply to the continuous state space model. However, these results are consistent with our analytic results; they suggest that the equilibrium is not unique when the amount of congestion is significant. The results may be of independent interest because the discrete state-space game may be as good an approximation of the environment as is the continuous state-space game.

4. Conclusion

We characterized the equilibrium set in an incomplete information global game where players' actions can be both strategic complements and strategic substitutes. We provided a necessary and sufficient condition for existence of the switching strategy equilibrium that is identified as the unique equilibrium in models where actions are always strategic complements. This condition requires that congestion is not too severe, compared to the noise in players' signals. An alternative view of this condition is that strategic uncertainty is not too great compared to fundamental uncertainty.

We showed that when this condition is not satisfied, agents' decisions are non-monotonic in the signal. If a pure strategy equilibrium exists and there is strong congestion, the length of intervals over which a particular action is optimal is linearly related to ϵ, the amount of noise. Thus, when the noise is small, the optimal action changes frequently as the signal varies. Any mixed strategy equilibrium must also be non-monotonic. These non-monotonic equilibria capture the spirit of Yogi Berra's aphorism that we quote at the beginning of this paper. In equilibrium, agents do not go to a bar, despite getting a more favorable signal about its fundamental quality, because they anticipate that too many others will be there.

We have not dealt with the uniqueness of equilibrium in the incomplete information game. The iterated deletion argument used for uniqueness in previous analyses of global games relies heavily on the fact that the game is supermodular when actions are strategic complements; see Morris and Shin (2003), Milgrom and Roberts (1990) and Vives (1990). Under some situations, uniqueness can be established even when there are congestion effects. For example, Goldstein and Pauzner (2002) establish the uniqueness of a switching equilibrium in a model of bank-runs. In their model there is an interval (of α) over which the benefit of taking a particular action is non-monotonic in the measure of agents who take the same action; however, for every realization of the market fundamental, the expected value of taking the action switches signs at most once. Our model does not satisfy this property. In our model, for *any* (positive) amount of congestion there are levels of the market fundamental for which the value of taking a particular action switches signs twice. Therefore, we do not expect to be able to establish uniqueness generally. In future work, we hope to establish conditions on the degree of congestion and signal precision that ensure equilibrium uniqueness.

Appendix A

The first part of the appendix proves the propositions and remarks, and the second part discusses the algorithm used to solve the discrete state space model.

A.1. Proofs

Proof of Proposition 1. We use the Glicksberg's fixed point theorem to establish the existence of equilibrium.

From Milgrom and Weber (1985, Theorem 1), each player's set of distributional strategies is a compact, convex metric space in the weak topology. In order to establish continuity of expected payoffs, we shall use the sup-norm metric on the set of distributional strategies. The topology induced by this metric is necessarily stronger than the weak topology, since the weak topology ensures continuity only of linear functionals. Hence each player's set of distributional strategies remains compact under this metric. The sup-norm metric is defined as follows. Let the set of distributional strategies for a player be Y. For any $y, y' \in Y$, define the metric

$$d(y, y') \equiv \sup_{x \in \mathbf{R}} \left| y(x) - y'(x) \right|.$$

Similarly, on the space Y^2, define the metric

$$\rho\big((y_0, y_1), (y_2, y_3)\big) \equiv \max\big\{d(y_0, y_1), d(y_2, y_3)\big\}.$$

Now consider any sequence $\{(y_i^n, y^n)\}$ in Y^2 that converges (under the metric ρ) to the limit $(y_i, y) \in Y^2$. Player i's expected payoff along this sequence is

$$u(x, y_i^n, y^n) = \left(\frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} \left(\theta + f\left(\frac{1}{2\epsilon} \int_{\theta-\epsilon}^{\theta+\epsilon} y^n(z)\, \mathrm{d}z \right) \right) \mathrm{d}\theta \right) y_i^n(x)$$

given the signal realization x. Therefore

$$\left| u(x, y_i, y) - u\big(x, y_i^n, y^n\big) \right|$$

$$= \left| x\big(y_i(x) - y_i^n(x)\big) + \frac{1}{2\epsilon}\left(y_i(x) \int_{x-\epsilon}^{x+\epsilon} f\left(\frac{1}{2\epsilon}\int_{\theta-\epsilon}^{\theta+\epsilon} y(z)\,\mathrm{d}z\right)\mathrm{d}\theta \right.\right.$$

$$\left.\left. - y_i^n(x) \int_{x-\epsilon}^{x+\epsilon} f\left(\frac{1}{2\epsilon}\int_{\theta-\epsilon}^{\theta+\epsilon} y^n(z)\,\mathrm{d}z\right)\mathrm{d}\theta \right)\right|$$

$$\leqslant \left| x\big(y_i(x) - y_i^n(x)\big)\right|$$

$$+ \left| \frac{1}{2\epsilon}\left(y_i(x) \int_{x-\epsilon}^{x+\epsilon} f\left(\frac{1}{2\epsilon}\int_{\theta-\epsilon}^{\theta+\epsilon} y(z)\,\mathrm{d}z\right)\mathrm{d}\theta - y_i^n(x) \right.\right.$$

$$\left.\left. \times \int_{x-\epsilon}^{x+\epsilon} f\left(\frac{1}{2\epsilon}\int_{\theta-\epsilon}^{\theta+\epsilon} y^n(z)\,\mathrm{d}z\right)\mathrm{d}\theta \right)\right|. \tag{8}$$

We need to consider only signal realizations in the interval $[\underline{\theta}, \bar{\theta}]$, since outside of this interval, the player has a dominant strategy (play 0 (1) with probability 1 for all signals $x < \underline{\theta} (> \bar{\theta})$). Hence

$$\left| x\big(y_i(x) - y_i^n(x)\big)\right| \leqslant \max\{|\underline{\theta}|, |\bar{\theta}|\} d\big(y_i, y_i^n\big).$$

Now consider the term on the second line of Eq. (8). It is less than or equal to $1/2\epsilon$ times

$$\left| \big(y_i(x) - y_i^n(x)\big) \int_{x-\epsilon}^{x+\epsilon} f\left(\frac{1}{2\epsilon}\int_{\theta-\epsilon}^{\theta+\epsilon} y(z)\,\mathrm{d}z\right)\mathrm{d}\theta \right|$$

$$+ \left| y_i^n(x)\left(\int_{x-\epsilon}^{x+\epsilon} f\left(\frac{1}{2\epsilon}\int_{\theta-\epsilon}^{\theta+\epsilon} y(z)\,\mathrm{d}z\right)\mathrm{d}\theta - \int_{x-\epsilon}^{x+\epsilon} f\left(\frac{1}{2\epsilon}\int_{\theta-\epsilon}^{\theta+\epsilon} y^n(z)\,\mathrm{d}z\right)\mathrm{d}\theta \right)\right|.$$

The first term is bounded above by $\bar{f} d(y_i, y_i^n)$. The second term is bounded above by

$$\left| \left(\int_{x-\epsilon}^{x+\epsilon} f\left(\frac{1}{2\epsilon}\int_{\theta-\epsilon}^{\theta+\epsilon} y(z)\,\mathrm{d}z\right)\mathrm{d}\theta - \int_{x-\epsilon}^{x+\epsilon} f\left(\frac{1}{2\epsilon}\int_{\theta-\epsilon}^{\theta+\epsilon} y^n(z)\,\mathrm{d}z\right)\mathrm{d}\theta \right)\right|$$

since $y_i^n \in [0, 1]$. Since $f(\cdot)$ is analytic, it can be represented by a Taylor series. Hence the last absolute value can be written as

$$\left| \int_{x-\epsilon}^{x+\epsilon} \left(\frac{1}{2\epsilon}\big(y(z) - y^n(z)\big) f'\left(\frac{1}{2\epsilon}\int_{\theta-\epsilon}^{\theta+\epsilon} y(z)\,\mathrm{d}z\right) + \frac{1}{4\epsilon^2}\big(y(z) - y^n(z)\big)^2 f''\left(\frac{1}{2\epsilon}\int_{\theta-\epsilon}^{\theta+\epsilon} y(z)\,\mathrm{d}z\right) \right.\right.$$

$$\left.\left. + \frac{1}{8\epsilon^3}\big(y(z) - y^n(z)\big)^3 f'''\left(\frac{1}{2\epsilon}\int_{\theta-\epsilon}^{\theta+\epsilon} y(z)\,\mathrm{d}z\right) + \cdots \right)\mathrm{d}\theta \right|.$$

All of the derivatives of $f(\cdot)$ are bounded (again, because f is analytic). Hence this is less than or equal to

$$\frac{1}{2\epsilon}d\big(y, y^n\big)\bar{f}' + \frac{1}{4\epsilon^2}\big(d\big(y, y^n\big)\big)^2\bar{f}'' + \frac{1}{8\epsilon^3}\big(d\big(y, y^n\big)\big)^3\bar{f}''' + \cdots$$

where e.g., $\bar{f}'$ is the bound on the first derivative of f.

Bringing these terms together,

$$\begin{aligned}\big|u(x, y_i, y) - u\big(x, y_i^n, y^n\big)\big| \leqslant{} & \max\big\{|\underline{\theta}|, |\bar{\theta}|\big\}d\big(y_i, y_i^n\big) + \frac{1}{2\epsilon}\bigg(\bar{f}d\big(y_i, y_i^n\big) \\ & + \frac{1}{2\epsilon}d\big(y, y^n\big)\bar{f}' + \frac{1}{4\epsilon^2}\big(d\big(y, y^n\big)\big)^2\bar{f}'' + \frac{1}{8\epsilon^3}\big(d\big(y, y^n\big)\big)^3\bar{f}''' + \cdots\bigg).\end{aligned}$$

Since the sequence $\{(y_i^n, y^n)\}$ converges to the limit (y_i, y), for any $\delta > 0$, there exists a finite N such that for any $n > N$, $|u(x, y_i, y) - u(x, y_i^n, y^n)| < \delta$. This establishes continuity of the expected payoff function.

Finally, the expected payoff is clearly quasi-concave (in fact, linear) in y_i. Hence, when distributional strategies are topologized by the sup-norm metric, the players' strategy sets are compact, convex metric spaces and the (expected) payoff functions are continuous and linear. By Glicksberg's theorem, an equilibrium exists. □

Proof of Lemma 2. From the definition of expected payoff,

$$u(x^*, S_{x^*}) = \frac{1}{2\epsilon}\int_{x^*-\epsilon}^{x^*+\epsilon}\big(\theta + f\big(\alpha(\theta, S_{x^*})\big)\big)\,\mathrm{d}\theta. \tag{9}$$

Conditional on receiving the signal x^*, the distribution of α is uniform on [0, 1]. Therefore

$$u(x^*, S_{x^*}) = x^* + \int_0^1 f(\alpha)\,\mathrm{d}\alpha.$$

Hence $u(x^*, S_{x^*}) = 0$ if and only if $x^* = -\int_0^1 f(\alpha)\,\mathrm{d}\alpha$. □

Proof of Proposition 2. The switching strategy around x^* is an equilibrium if and only if $u(x, S_{x^*})$ is single upward crossing: that is, (i) $u(x, S_{x^*}) < 0$ for $x < x^*$; (ii) $u(x, S_{x^*}) > 0$ for $x > x^*$. We show that these two conditions are equivalent to condition (2).

From the definition of the expected payoff and the equilibrium proportion of individuals who choose 'go', from Eq. (1),

$$\begin{aligned} u(x, S_{x^*}) &= x + \frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} f\big(\alpha(\theta, S_{x^*})\big)\,\mathrm{d}\theta \\ &= \begin{cases} x & \text{if } x < x^* - 2\epsilon, \\ x + \frac{1}{2\epsilon}\int_{x^*-\epsilon}^{x+\epsilon} f(\frac{1}{2} + \frac{1}{2\epsilon}(\theta - x^*))\,\mathrm{d}\theta & \text{if } x^* - 2\epsilon \leqslant x < x^*, \\ x + \frac{1}{2\epsilon}\int_{x-\epsilon}^{x^*+\epsilon} f(\frac{1}{2} + \frac{1}{2\epsilon}(\theta - x^*))\,\mathrm{d}\theta + \frac{1}{2\epsilon}\int_{x^*+\epsilon}^{x+\epsilon} f(1)\,\mathrm{d}\theta & \text{if } x^* \leqslant x < x^* + 2\epsilon, \\ x + f(1) & \text{if } x \geqslant x^* + 2\epsilon. \end{cases} \end{aligned} \tag{10}$$

From Lemma 2, $x^* = -\int_0^1 f(\alpha)\,\mathrm{d}\alpha$. Let $z \equiv \frac{x-x^*}{2\epsilon}$. We first consider the case where $x < x^*$ and then the case where $x > x^*$.

Using the first line of Eq. (10), $u(x, S_{x^*}) < 0$ for $x < x^* - 2\epsilon$, if and only if $x^* - 2\epsilon < 0$ i.e.,

$$2\epsilon > -\int_0^1 f(\alpha)\,\mathrm{d}\alpha. \tag{11}$$

For $x \in [x^* - 2\epsilon, x^*]$, the second line of Eq. (10) implies

$$u(x, S_{x^*}) = 2\epsilon z + x^* + \int_0^{z+1} f(\alpha)\,\mathrm{d}\alpha. \tag{12}$$

Using Eq. (12) and the definition of x^*, we write the equilibrium condition $u(x, S_{x^*}) < 0$ as

$$2\epsilon z - \int_0^1 f(\alpha)\,\mathrm{d}\alpha + \int_0^{z+1} f(\alpha)\,\mathrm{d}\alpha = 2\epsilon z - \int_{z+1}^1 f(\alpha)\,\mathrm{d}\alpha < 0$$

for all $z \in (-1, 0)$. Since z is a negative number, the last inequality is equivalent to

$$2\epsilon \geqslant \frac{1}{z}\int_{z+1}^1 f(\alpha)\,\mathrm{d}\alpha \tag{13}$$

for all $z \in (-1, 0)$. Inequalities (11) and (12) are equivalent to

$$2\epsilon \geqslant \max_{z\in[-1,0)}\left(\frac{1}{z}\int_{z+1}^1 f(\alpha)\,\mathrm{d}\alpha\right). \tag{14}$$

We now consider the case where $x > x^*$. From the last line of Eq. (10), $u(x, S_{x^*}) > 0$ for $x > x^* + 2\epsilon$, if and only if $x^* + 2\epsilon + f(1) > 0$, i.e.,

$$2\epsilon > \int_0^1 f(\alpha)\,\mathrm{d}\alpha - f(1). \tag{15}$$

Using the third line of line of Eq. (10) and the definition of z, we have $u(x, S_{x^*}) > 0$ for $x^* \leqslant x < x^* + 2\epsilon$, if and only if $x^* + 2\epsilon z + \int_z^1 f(\alpha)\,\mathrm{d}\alpha + zf(1) > 0$ for $z \in [0, 1)$. Using the definition of x^*, we rewrite this inequality as

$$2\epsilon > \frac{1}{z}\int_0^z f(\alpha)\,\mathrm{d}\alpha - f(1) \tag{16}$$

for $z \in [0, 1)$. Inequalities (15) and (16) are equivalent to

$$2\epsilon > \max_{z\in[0,1]}\left(\frac{1}{z}\int_0^z f(\alpha)\,\mathrm{d}\alpha - f(1)\right). \tag{17}$$

Inequalities (14) and (17) are both satisfied if and only if Eq. (2) is satisfied. □

Proof of Remark 1. If the inequality

$$\frac{\partial}{\partial x} u(x, S_{x^*}) > 0 \tag{18}$$

holds for *arbitrary* x^*, then the switching strategy with switch point x^* specified in Lemma 2 is an equilibrium. At this value of x^*, agents are indifferent between the two actions, and if inequality (18) holds, they strictly prefer to go to the bar (not go to the bar) when $x > x^*$ ($x < x^*$). Thus it is sufficient to show that inequality (18) holds for arbitrary x^*.

Using Eq. (10), we have

$$\frac{\partial u(x, S_{x^*})}{\partial x} = \begin{cases} 1 & \text{if } x < x^* - 2\epsilon, \\ 1 + \frac{1}{2\epsilon} f(\frac{1}{2} - \frac{1}{2\epsilon}(x - x^* + \epsilon)) & \text{if } x^* - 2\epsilon \leqslant x < x^*, \\ 1 - \frac{1}{2\epsilon} f(\frac{1}{2} - \frac{1}{2\epsilon}(x - x^* - \epsilon)) + \frac{1}{2\epsilon} f(1) & \text{if } x^* \leqslant x < x^* + 2\epsilon, \\ 1 & \text{if } x \geqslant x^* + 2\epsilon. \end{cases} \tag{19}$$

From the third line, $\frac{\partial}{\partial x} u(x, S_{x^*}) > 0$ if $f(1) + 2\epsilon > f(\alpha)$ for all $\alpha \in [0, 1]$. This relation holds if and only if $f(1) \geqslant \overline{f} - 2\epsilon$. From the second line of Eq. (19), $\frac{\partial}{\partial x} u(x, S_{x^*}) > 0$ if $f(\alpha) > -2\epsilon$ for all $\alpha \in [0, 1]$. By Assumption 2(iv), this relation holds if and only if $f(1) > -2\epsilon$, which is weaker than the condition $f(1) \geqslant \overline{f} - 2\epsilon$. Thus, the latter inequality is equivalent to the requirement that condition (18) holds for arbitrary x^*.

Proof of Remark 2. If $-\hat{\theta} \equiv f(1) < \int_0^1 f(\alpha)\,\mathrm{d}\alpha \equiv -x^*$, then $x^* < \hat{\theta}$. Let $\bar{\epsilon} = (\hat{\theta} - x^*)/2 > 0$, and suppose that $\epsilon < \bar{\epsilon}$. Given such an ϵ, the individual who receives a private signal of $x \in (x^* + 2\epsilon, \hat{\theta})$ knows with certainty that $\alpha = 1$, since $x > x^* + 2\epsilon$. The expected payoff from choosing 'go' is therefore $x + f(1) < 0$, since $x < \hat{\theta}$. It follows that the monotonic switching strategy around x^* cannot be an equilibrium. □

Proof of Proposition 3. In order to show that no island is isolated, suppose that at least one island is isolated. Denote the lower and upper boundaries of this island by $\underline{x}$ and $\bar{x}$. The assumption that the island is isolated implies that the distribution of α is symmetric at the two boundaries. This fact implies that $\mathbb{E}(f(\alpha) \mid \underline{x}) = \mathbb{E}(f(\alpha) \mid \bar{x})$. This equality is not consistent with the equilibrium condition that at the boundary of an island the agent is indifferent between 'go' and 'don't go', i.e., $\mathbb{E}(\theta + f(\alpha) \mid \underline{x}) = 0 = \mathbb{E}(\theta + f(\alpha) \mid \bar{x})$ and $\underline{x} < \bar{x}$. Thus, the island cannot be isolated.

For the second part of the proposition we use the fact that all islands lie in the non-dominance region, so any signal on an island, $x^\#$, satisfies

$$x^\# + \max_\alpha f(\alpha) > 0 > x^\# + \min_\alpha f(\alpha).$$

If $f(1) < 0$, then Assumption 1 implies that $\min_\alpha f(\alpha) = f(1)$, so $0 > x^\# + f(1)$. Suppose, contrary to the proposition, that $f(1) < 0$ and that the length of some island is greater than 4ϵ. The agent who receives a signal $x^\#$ near the midpoint of this island is certain that all other agents received signals on the island, and therefore is certain that $\alpha = 1$. Since the island is in the non-dominance region, $0 > x^\# + \min_\alpha f(\alpha) = x^\# + f(1)$. It is not an equilibrium for the agent with signal $x^\#$ to choose 'go'. Thus, the island cannot be longer than 4ϵ.

If $f(1) > 0$, then Assumption 1 implies $\min_\alpha f(\alpha) = 0$. Suppose that for $f(1) > 0$ and that there are two adjacent signal intervals, each longer than 4ϵ, for which it is an equilibrium to chose 'go'. Denote x_1 as the upper boundary of the lower interval and x_2 as the lower boundary of the upper interval. Since each interval is longer than 4ϵ, the distribution of α is symmetric at x_1

and x_2, so $\mathbb{E}(f(\alpha) \mid x_1) = \mathbb{E}(f(\alpha) \mid x_2)$. Since agents are indifferent at x_1 and x_2. $\mathbb{E}(\theta + f(\alpha) \mid x_1) = 0 = \mathbb{E}(\theta + f(\alpha) \mid x_2)$. The previous two equalities contradict the fact that $x_1 < x_2$.

In order to show that no island is isolated, suppose that at least one island is isolated. Denote the lower and upper boundaries of this island by $\underline{x}$ and $\bar{x}$. The assumption that the island is isolated implies that the distribution of α is symmetric at the two boundaries. This fact implies that $\mathbb{E}(f(\alpha) \mid \underline{x}) = \mathbb{E}(f(\alpha) \mid \bar{x})$. This equality is not consistent with the equilibrium condition that at the boundary of an island the agent is indifferent between 'go' and 'don't go', i.e., $\mathbb{E}(\theta + f(\alpha) \mid \underline{x}) = 0 = \mathbb{E}(\theta + f(\alpha) \mid \bar{x})$ and $\underline{x} < \bar{x}$. Thus, the island cannot be isolated.

In order to show that the length of the island is less than 4ϵ, suppose that it is greater than 4ϵ. The agent who receives a signal near the midpoint of this island is certain that all other agents received signals on the island, and therefore is certain that $\alpha = 1$. Since the island is in the non-dominance region, it is not an equilibrium for this agent to choose 'go'. Thus, the island cannot be longer than 4ϵ.

Proof of Proposition 4. The proof of the proposition has two steps. In the first, we show that if $f(1) \geqslant 0$, then there is no equilibrium with a single randomization interval. In the second, we show that if $f(1) < 0$, then any equilibrium with a single randomization interval is non-monotonic.

If $f(1) \geqslant 0$, then, by Assumption 2(iv) $f(\alpha) \geqslant 0$ for all $\alpha \in [0, 1]$. Suppose that there is a mixed strategy equilibrium with a single randomization interval. The equilibrium condition (6) implies that $f(\alpha(\underline{x}) + \epsilon)) = f(\alpha(\underline{x}) - \epsilon)) - 2\epsilon$. Moreover $f(\alpha(\underline{x}) - \epsilon)) = f(0) = 0$. Hence $f(\alpha(\underline{x}) + \epsilon)) = -2\epsilon$, which is impossible if $f(\alpha) \geqslant 0$ for all $\alpha \in [0, 1]$. It follows that there is no mixed strategy equilibrium with a single randomization interval.

Now consider the case that $f(1) < 0$. Consider a monotonic mixed strategy $y_M(\cdot)$ with a single randomization interval $[\underline{x}, \bar{x}]$ played by all agents. We shall show that the function $\alpha(\theta, y_M)$ induced by this strategy is non-monotonic. Since this is contrary to the assumption that $y_M(\cdot)$, we prove by contradiction that any mixed strategy with a single randomization interval must be non-monotonic.

Define $\phi(x) \equiv f \circ \alpha$, the composition of the functions $f(\alpha)$ and $\alpha(x)$. From the definition of ϕ:

$$\phi(x) = f\big(\alpha(x)\big), \tag{20}$$

$$\phi'(x) = f'\big(\alpha(x)\big)\alpha'(x) \tag{21}$$

where the derivatives exist, i.e., everywhere except for at most a finite number of points (since $y_M(\cdot)$ is monotonic—see the argument in Section 3.4).

By Assumption 2, $f(\alpha)$ has a single peak and it is increasing at $\alpha = 0$. Therefore: (i) $f(\alpha) > 0$ for α small, and (ii) $f(\alpha) < 0$ implies $f'(\alpha) < 0$. Since $\alpha(\underline{x} - \epsilon) = 0$, α is continuous, and is positive on the mixing interval, it must be increasing in the neighborhood to the right of $\underline{x} - \epsilon$. Therefore ϕ is increasing in the neighborhood to the right of $\underline{x} - \epsilon$; that is, $\phi(\underline{x} - \epsilon + \delta)$ is increasing in δ for small positive δ. From Eq. (6), $\phi(\underline{x} + \epsilon + \delta) = \phi(\underline{x} - \epsilon + \delta) - 2\epsilon$. Consequently ϕ is increasing in the neighborhood to the right of $\underline{x} + \epsilon$.

Due to the fact that $\phi(\underline{x} - \epsilon) = f(0) = 0$ and the equilibrium condition (6), $\phi(\underline{x} + \epsilon) = -2\epsilon$. Therefore $f(\alpha(\underline{x} + \epsilon)) < 0$. (This inequality is possible because we consider only the case $f(1) < 0$.) Therefore $f'(\alpha(\underline{x} + \epsilon)) < 0$. By continuity of α, $f' < 0$ in the neighborhood to the right of $\underline{x} + \epsilon$.

Since $\phi' > 0$ and $f' < 0$ in the neighborhood to the right of $\underline{x} + \epsilon$, we conclude that $\alpha'(x) < 0$ in that neighborhood (at points where the derivative exists). We noted above that α is increasing in the neighborhood to the right of $\underline{x} - \epsilon$. Thus, α is non-monotonic. □

A.2. Details of numerical methods

We used two algorithms. The simple algorithm restricts strategies to be pure and begins with the guess of the switching strategy discussed in Section 3.2. Using this guess, we calculate the induced $\alpha(\theta)$, and then find the equilibrium response. Using this response as our next guess, we iterate. This algorithm converges quickly to a pure strategy island equilibrium, or it fails to converge—even after hundreds of thousands of iterations.

The more elaborate algorithm finds a mixed strategy equilibrium by solving a nonlinear complementarity problem. Define p as the N dimensional vector of probabilities, with p_i equal to the probability that an agent who observes signal i goes to the bar. The vector p induces a conditional distribution $\alpha(\theta_i)$ which is used to calculate $u(i, p)$, the expected benefit of going to the bar for an agent who observes signal i, when the strategy profile is p. The equilibrium condition is

$$\begin{aligned} &p_i > 0 \Rightarrow u(p,i) \geqslant 0, \\ &p_i < 1 \Rightarrow u(p,i) \leqslant 0. \end{aligned} \tag{22}$$

The following discussion relies on Chapter 3.8 of Miranda and Fackler (2002), and we use the Matlab toolbox (particularly, the program "ncpsolve") that accompanies their book to implement the algorithm. A vector p solves the complementarity problem (22) if and only if it solves the root-finding problem

$$\hat{u}(p) = \min\bigl(\max\bigl(u(p), 0\varrho - p\bigr), 1\varrho - p\bigr) = 0, \tag{23}$$

where $u(p)$ is the vector-valued function whose ith element is $u(p, i)$, and ϱ is an N-dimensional vector of ones. The proof of the equivalence between Eqs. (22) and (23) requires the enumeration of the possibilities implied by the min and max operators in Eq. (23).

The function $\hat{u}$ has points of non-differentiability, which may make it difficult to find a solution using Newton's method. This problem can be avoided by using Fischer's function, defined as

$$\tilde{u}(p) = \phi^-\bigl(\phi^+\bigl(u(p), 0\varrho - p\bigr), 1\varrho - p\bigr)$$

where

$$\phi_i^{\pm}(x, y) = x_i + y_i \pm \sqrt{x_i^2 + y_i^2}.$$

The values of the functions $\hat{u}(p)$ and $\tilde{u}(p)$ may differ significantly, but they have the same signs, and the same roots. The function $\tilde{u}$ is smoother, making it easier to find a root using Newton's method. We solve the original problem by finding the roots to $\tilde{u}(p) = 0$

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